REALIZATION OF SATURATED TRANSFER SYSTEMS ON CYCLIC GROUPS OF ORDER $p^n q^m$ BY LINEAR ISOMETRIES N_∞-OPERADS

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Abstract. We prove a new case of Rubin's saturation conjecture about the realization of G-transfer systems, for G a finite cyclic group, by linear isometries N_∞-operads, namely the case of cyclic groups of order $p^n q^m$ for p, q distinct primes and $n, m \in \mathbb{N}$.

Contents

1. Introduction

Blumberg and Hill study in [BH15] the question of generalizing the theory of E_∞-operads to the G-equivariant setting, for a finite group G. Instead of considering E_∞-operads in categories of G-objects, their approach is to take into account the G-action at a finer level, by considering operads in G-spaces, with contractibility conditions on the subspaces of fixed points in the space of operations in arity n, for all subgroups of $G \times S_n$, for S_n the symmetric group (in particular, they must be either contractible or empty). The concept obtained is that of N_∞-operad, and represents (a whole range of) intermediary notion(s) between E_∞-operads in a category and E_∞-operads in its category of G-objects.

A morphism between such operads is a weak equivalence if it induces weak equivalences of topological spaces in all arities, on all subspaces of fixed points for subgroups of $G \times S_n$.

An N_∞-operad imposes on its algebras the existence of additional structure: contractible spaces of transfer maps, and the information of which transfers exist is parametrized by the data of which spaces of fixed points in the operad are contractible, and not empty.

Date: October 2023.

These N_∞-operads are classified up to homotopy by subposets of the poset of subgroups of G satisfying additional conditions (namely being closed under conjugation and restriction), the so-called *G-transfer systems*.

Theorem 2.9 (3.24 in [BH15]; and [BP21], [GW18], [Rub21a]). *The homotopy category of N_∞-operads is equivalent to the poset of G-transfer systems ordered by refinement.*

In their foundational article, Blumberg and Hill defined a functor between these categories and showed its fullness and faithfulness, while (essential) surjectivity was only proved some years later (Bonventre & Pereira ([BP21]), Gutiérrez & White ([GW18]), Rubin ([Rub21a])).

Section 2 provides a brief introduction to N_∞-operads and their classification.

A particular class of N_∞-operads is provided by linear isometries operads. Given a G-universe $\mathcal{U}$, which is a countably infinite-dimensional real representation of G by linear isometries, with all sub-representations occurring infinitely many times, there is an N_∞-operad $\mathcal{L}(\mathcal{U})$ associated with it, whose operations in arity n consist in the space of isometries $\mathcal{U}^{\oplus n} \to \mathcal{U}$. A realization problem then arises: which G-transfer systems can be realized by linear isometries operads, as the G-universe varies? Blumberg and Hill found a necessary condition: the transfer system must be saturated (if it contains the relation $H \subseteq K$ for some subgroups H and K of G, then for any intermediary subgroup $H \subseteq M \subseteq K$, the transfer system must also contain the relations $H \subseteq M$ and $M \subseteq K$). This condition may however not be sufficient, as proved in [Rub21b], and as we recall in Remark 3.7.

Rubin conjectured in [Rub21b] that in the case of finite cyclic groups, the saturation condition might actually become sufficient.

Conjecture 3.4 (Rubin's saturation conjecture). *Let $k \in \mathbb{N}^*$ and $e_1, \ldots, e_k \in \mathbb{N}^*$. There exist integers $p_1, \ldots, p_k$ depending on this choice, such that for all k-uples of distinct primes $q_1, \ldots, q_k$ with $q_i \geq p_i$ for all $i \leq k$, and $G := C_{q_1^{e_1} \cdots q_k^{e_k}}$, any saturated G-transfer system is realized by some linear isometries operad.*

The conjecture was proved in [Rub21b] for cyclic groups of order p^n and pq, and in [HMOO22] for cyclic groups of order qp^n, for $p, q \geq 5$ distinct primes and any $n \in \mathbb{N}$. For arbitrary cyclic groups, the problem was reduced to a purely arithmetic one by Rubin in [Rub21b]: there is a characterization of the relations contained in the transfer system arising from $\mathcal{L}(\mathcal{U})$, in terms of the translations under which a certain subset of C_n, called the *indexing set*, that uniquely characterizes the universe $\mathcal{U}$, is invariant. We discuss this conjecture, and the reduction of the problem to modular arithmetic in section 3.

The proofs of the three cases of the conjecture mentioned above consist in building suitably invariant indexing sets. Following the same approach, and using the previously proven cases as a basis for induction, we prove in section 4 our main result, the following new instances of the saturation conjecture.

Theorem 3.8 (Saturation conjecture for $p^n q^m$). *Let $p, q \geq 5$ be distinct primes, and $n, m \in \mathbb{N}$. If $G = C_{p^n q^m}$, then any saturated G-transfer system is realized by some linear isometries operad on a G-universe $\mathcal{U}$.*

This result does not hold for $p \leq 3$ or $q \leq 3$, see Remark 3.7 for a counterexample. Saturated transfer systems on $C_{p^nq^m}$ were enumerated in [HMOO22], by describing them in terms of *saturated covers* of $[m] \times [n]$ and *compatible codes*.

Acknowledgements. The work presented in this paper was carried out during a summer internship in the Laboratory for Topology and Neuroscience at EPFL. I would like to thank the EPFL "Summer in the Lab" and "Student Support" programs for making this possible and the "Domaine de Villette" foundation for their support to the programs. I also would like to express all my gratitude to K. Hess Bellwald and J. Scherer for their continued encouragement and kind guidance.

2. G-equivariant analog to E_∞-operads : N_∞-operads

We give here the formal definition of the equivariant operads advertised in the introduction and present their classification up to homotopy by transfer systems. In this section, let G be a fixed finite (discrete) group.

Definition 2.1. A *G-operad* is a topological operad $\mathcal{O}$ (or an operad in G-spaces), such that $\mathcal{O}(n)$ is a $(G \times S_n)$-space for all $n \in \mathbb{N}$, with a G-fixed unity and G-equivariant structure maps. An *N_∞-operad* is a G-operad such that the following conditions hold:

- For all $n \in \mathbb{N}$, the action of the symmetric group S_n on $\mathcal{O}(n)$ is free.
- For all subgroups $\Gamma \leq G \times S_n$, the space $\mathcal{O}(n)^\Gamma$ of Γ-fixed points is either empty or contractible (as a topological space, not necessarily equivariantly).
- The space of fixed points $\mathcal{O}(n)^G$ is non-empty for all $n \in \mathbb{N}$.

2.1. Algebras over N_∞-operads.

We clarify what we mean by algebras over such operads. Let Top be the category of topological spaces, and ${}_G\mathsf{Top}$ the category of G-spaces.

Definition 2.2. Given $\mathcal{O}$ an N_∞-operad, an *algebra in ${}_G\mathsf{Top}$ over $\mathcal{O}$* is an algebra over the underlying operad in G-spaces.

In fact, N_∞-algebras can be defined in any symmetric monoidal category that is tensored over the category of G-spaces. In [BH15] the case of orthogonal G-spectra is studied in detail. We shall work only with topological spaces in this article.

The specific axioms of N_∞-operads ensure the existence of additional structure for their algebras, namely *transfers maps* (*norm maps* for spectra). The notion of an admissible relation is defined at the end of this section in 2.11 and 2.12.

Theorem 2.3 (7.1 and 7.2 in [BH15], 3.5 in [Rub21b])**.** *Let $\mathcal{O}$ be an N_∞-operad, and X an algebra in ${}_G\mathsf{Top}$ over $\mathcal{O}$.*

- *Given an admissible relation $K \to H$ for subgroups $K \leq H \leq G$, there is a contractible space of maps $(G \times S_{[H:K]})/\Gamma_{H,K} \to \mathcal{O}([H:K])$.*
- *Again assuming that $K \to H$ is admissible, there are contractible spaces of* internal transfer maps *of H-spaces $X^K \longrightarrow X^H$ and* external transfer maps *of G-spaces $G \times_H X^{\times H/K} \longrightarrow X$.*
- *If $K \leq H$ and $N \leq H$ are admissible, any H-equivariant map $H/K \longrightarrow H/N$ induces a contractible space of maps:* $\mathsf{Top}(H/K, X) \longrightarrow \mathsf{Top}(H/N, X)$*.*

Weak equivalences between N_∞-operads should also respect the finer structure of all spaces of fixed points. Otherwise, they would amount only to weak equivalences of the underlying topological E_∞-operads. But all categories of algebras over such operads are equivalent, whereas for different N_∞-operads the algebras do not look quite the same, due to the transfer maps. The richer structure on N_∞-operads becomes apparent in their classification up to homotopy in Theorem 2.9.

Definition 2.4. A morphism of N_∞-operads $f : \mathcal{O} \to \mathcal{O}'$ is called a *weak equivalence* if the underlying map of G-operads is a weak equivalence, namely if the induced G-equivariant map $f^\Gamma : \mathcal{O}(n)^\Gamma \to \mathcal{O}'(n)^\Gamma$ is a (non-equivariant) weak homotopy equivalence on the underlying topological spaces for all $n \in \mathbb{N}$ and all subgroups $\Gamma \leq G \times S_n$. By localizing with respect to these maps (as a category with weak equivalences), we obtain the *homotopy category* of N_∞-operads, denoted by $\mathrm{Ho}(N_\infty\text{-Op})$.

Definition 2.4 is justified by the fact that weakly equivalent N_∞-operads have Quillen equivalent categories of algebras, provided that $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ are nice enough spaces for all $n \in \mathbb{N}$ (see [BH15], Theorem A.3).

2.2. **Linear isometries operads.** To define this class of N_∞-operads we are particularly interested in, we need specific representations of G.

Definition 2.5. A *G-universe* $\mathcal{U}$ is a real vector space of countably infinite dimension, endowed with an inner product and an action of G by linear isometries, such that any sub-representation occurs infinitely often, including the trivial one.

Since G is finite, any G-universe can be written as $\bigoplus_{\mathbb{N}}(\mathbb{R}_{\text{triv}} \oplus V_1 \oplus \cdots \oplus V_k)$ for some finite dimensional (irreducible) real linear isometric representations $V_1, \ldots, V_k$ of G, with $\mathbb{R}_{\text{triv}}$ the trivial representation.

Definition 2.6. Given $\mathcal{U}$ a G-universe, the *linear isometries operad* $\mathcal{L}(\mathcal{U})$ is the topological operad given in arity $n \in \mathbb{N}$ by the space of (non necessarily G-equivariant) isometries $\mathcal{U}^{\oplus n} \to \mathcal{U}$. The (left-) action of S_n and the composition maps are defined as in the usual endomorphism operad, with the unit being the identity on $\mathcal{U}$. Given an isometry $f : \mathcal{U}^{\oplus n} \to \mathcal{U}$, $g \in G$ and $(\vec{x}_1, \ldots, \vec{x}_n) \in \mathcal{U}^{\oplus n}$, define $(g \cdot f)(\vec{x}_1, \ldots, \vec{x}_n) = g \cdot f(g^{-1} \cdot \vec{x}_1, \ldots, g^{-1} \cdot \vec{x}_n)$ (action by conjugation).

In particular, a G-fixed point in $\mathcal{L}(\mathcal{U})(n)$ is just a G-equivariant linear isometry $\mathcal{U}^{\oplus n} \to \mathcal{U}$.

Other examples of N_∞-operads include the *equivariant little disks operads*, the *Steiner operads* or the *embeddings operads*, all depending on a G-universe (see [BH15]).

2.3. **Classification of N_∞-operads up to homotopy.** Blumberg and Hill provide in [BH15] a beautiful classification of N_∞-operads up to homotopy: their homotopy category is equivalent to a specific poset. To define the latter, we need to define transfer systems.

Definition 2.7. A *G-transfer system* is a relation $\to$ refining the inclusion $\leq$ in the lattice of subgroups of G, with, for all $K \leq H \leq G$:
- Reflexivity: $H \to H$.
- Transitivity: if $K \to M$ and $M \to H$ for some $M \leq G$, then $K \to H$.
- Closure under conjugation: if $K \to H$, then $(gKg^{-1}) \to (gHg^{-1})$ for all $g \in G$.
- Closure under self-induction/restriction: $K \to H \implies \forall M \leq H, K \cap M \to M$.

In other terms a G-transfer system is a full subcategory of the poset of subgroups of G, closed under conjugation and base change in pullback squares. Transfer systems on G form a poset $\mathrm{Tr}(G)$ with respect to inclusion (refinement).

Remark 2.8. When G is abelian, the conjugation condition holds trivially, so the definition does not use the group structure anymore and becomes purely combinatorial. It can therefore be generalized to any lattice, with intersection replaced by the meet operation. On finite lattices, transfer systems are in one-to-one correspondence with weak factorization systems (providing the right class of maps) and therefore with contractible model structures (i.e., with all maps being weak equivalences) (see [BOOR21]).

We are now ready to state the classification theorem.

Theorem 2.9 (3.24 in [BH15] and e.g. [Rub21a])**.** *There is an equivalence of categories*

$$\underline{\mathcal{C}} : Ho(N_\infty\text{-}Op) \longrightarrow Tr(G)$$

between the homotopy category of N_∞-operads and the poset of G-transfer systems.

Actually, Blumberg and Hill originally used another poset, the poset of *indexing systems*, to prove that $\underline{\mathcal{C}}$ was fully faithful, and conjectured it was an equivalence of categories, which was soon proved by several authors in different ways. Gutiérrez and White ([GW18]) for instance prove the existence of an N_∞-operad associated with each suitable sequence of subgroups of $G \times S_n$, using model structures on the category of G-operads. Bonventre and Pereira ([BP21]) use "genuine equivariant operads" and bar constructions. The proof by Rubin ([Rub21a]) is more combinatorial, and uses a "discrete" version of N_∞-operads, namely some particular operads in G-sets, whose homotopy theory is equivalent to that of N_∞-operads.

Using transfer systems instead of indexing systems constitutes an equivalent approach to the problem. The former can be seen as *generating data* for the latter: indexing systems are expressed in terms of categories of H-sets when H varies among the subgroups of G, and transfer systems correspond to the orbit objects in these categories, i.e., the H-sets isomorphic to H/K for $K \leq H$ a subgroup. Transfer systems can be more convenient to work with as they are smaller, with an a priori simpler definition.

Let us now describe the functor $\underline{\mathcal{C}}$. We need preliminary definitions.

Definition 2.10. Let $H \leq G$ be a subgroup and T a finite H-set. The *graph subgroup associated with* T is the (conjugacy class of the) subgroup $\Gamma_T \leq G \times S_{|T|}$ given by the graph of the homomorphism $H \to S_{|T|}$ sending $h \in H$ to the permutation $\sigma_h \in S_{|T|}$ such that $h \cdot t_i = t_{\sigma_h(i)}$ for all $i \leq |T|$, for $t_1, \dots, t_{|T|}$ an enumeration of T. If $T = H/K$ for some $K \leq H$, we write $\Gamma_T = \Gamma_{H,K}$.

This is an abuse of notation because this subgroup may change depending on the enumeration of T chosen. However, if we relabel T using a permutation τ, then the subgroups obtained are conjugated by τ. Actually all subgroups of $G \times S_n$ intersecting S_n trivially are graph subgroups, which can be seen as follows. Let $\Gamma \leq G \times S_n$ be such a subgroup, and $H \leq G$ be its projection on the first component. If $h \in H$, then there exists some $\sigma_h \in S_n$ with $(h, \sigma_h) \in \Gamma$. Moreover σ_h is unique with this property: if $(h, \tau) \in \Gamma$, then $(e, \sigma_h\tau^{-1}) \in \Gamma \cap S_n = \{(1, \mathrm{id})\}$, so $\sigma_h = \tau$. The assignment $h \mapsto \sigma_h$ is a group homomorphism: given $h, h' \in H$, we have

$(h,\sigma),(h',\sigma_{h'})\in\Gamma$, so $(hh',\sigma_h\sigma_{h'})\in\Gamma$, and by uniqueness $\sigma_{hh'}=\sigma_h\sigma_{h'}$. Therefore any such subgroup determines a morphism $H\to S_n$, which yields exactly an H-set structure on $\{1,\dots,n\}$.

Definition 2.11. Let $\mathcal{O}$ be an N_∞-operad and $H\leq G$ a subgroup. A finite H-set T is called *admissible* if $\mathcal{O}(|T|)^{\Gamma_T}\neq\emptyset$.

This is well-defined, because the condition about the fixed points being non-empty depends only on the conjugacy class of Γ_T.

Definition 2.12. The functor $\underline{\mathcal{C}}$ in Theorem 2.9 is induced, using the universal property of localization, by the functor $\underline{\mathcal{C}}$: N_∞-Op $\to$ Tr(G) sending an N_∞-operad $\mathcal{O}$ to the G-transfer system $\to$ such that $K\to H$ if and only if $K\leq H\leq G$ are subgroups, and H/K is admissible as an H-set for $\mathcal{O}$. The relations contained in the transfer system are called *admissible*.

3. Rubin's saturation conjecture

We present in this section Rubin's conjecture on saturated transfer systems for cyclic groups and his description of universes by indexing sets. For any $n\in\mathbb{N}^*$ and $m\mid n$, let $C_n:={}^{\mathbb{Z}}/_{n\mathbb{Z}}$ and $mC_n:={}^{m\mathbb{Z}}/_{n\mathbb{Z}}$. In particular we use additive notation. We repeat the proofs of Propositions 3.1 and 3.3 below only for the sake of completeness, and to add details.

By Theorem 2.9, all G-transfer systems are realized by some N_∞-operad. Blumberg and Hill asked which ones could be realized by a linear isometries operad. The admissible relations for the latter can be characterized as follows.

Proposition 3.1 (4.18 in [BH15])**.** *Given $K\leq H\leq G$, H/K is admissible for $\mathcal{L}(\mathcal{U})$ if and only if there exists an H-equivariant embedding $\mathbb{Z}[H/K]\otimes\mathcal{U}\to\mathcal{U}$.*

Remark 3.2. The group H acts on the tensor product $\mathbb{Z}[H/K]\otimes\mathcal{U}$ as follows: if $H/K=\{h_1K,\dots,h_kK\}$, and $h\in H$, $i\leq k$, $u\in\mathcal{U}$, there exists a unique $k_i(h)\in K$ with $hh_i=h_{\sigma_h(i)}k_i(h)$. We set $h\cdot(h_iK)\otimes u=h_{\sigma_h(i)}K\otimes k_i(h)u$. In particular, $\mathbb{Z}[H/K]\otimes\mathcal{U}$ is isomorphic as a representation of H to $\mathbb{Z}[H]\otimes_{\mathbb{Z}[K]}\mathcal{U}$.

Proof. Let us write $H/K=\{h_1K,\dots,h_nK\}$. A $\Gamma_{H,K}$-fixed point in $\mathcal{L}(\mathcal{U})(n)$ is by definition a $\Gamma_{H,K}$-equivariant map $F:\mathcal{U}^{\oplus n}\to\mathcal{U}$, where elements of the symmetric group permute the variables, and H acts by conjugation. By definition $\Gamma_{H,K}$ consists of elements (h,σ_h), where σ_h describes the permutation induced on H/K by $h\in H$. In particular, this fixed point gives us an H-equivariant embedding under the identification $\mathbb{Z}[H/K]\otimes\mathcal{U}\cong\mathcal{U}^{\oplus n}$, with $h_iK\otimes u$ sent to $h_i\cdot u$ on the i-th summand. Indeed, the map $f:\mathbb{Z}[H/K]\otimes\mathcal{U}\to\mathcal{U}$ obtained is H-equivariant. For all $h\in H$, $i\leq n$ and $u\in\mathcal{U}$, we obtain from the fixed point condition:

$$\begin{aligned}
h\cdot(f(h_iK\otimes u)) &= h\cdot(F(0,\dots,h_iu,\dots,0)) && (h_iu \text{ in the } i\text{-th summand})\\
&= h\cdot((h^{-1},\sigma_h^{-1})\cdot F)(0,\dots,h_iu,\dots,0)\\
&= hh^{-1}\cdot(F(0,\dots,hh_iu,\dots,0)) && (hh_iu \text{ in the } \sigma_h(i)\text{-th summand})\\
&= f(h_{\sigma_h(i)}K\otimes k_i(h)u)=f(h\cdot(h_iK\otimes u)).
\end{aligned}$$

□

Going back to our realization problem, the following condition is necessary.

Proposition 3.3 ([BH15]). *If G is a finite group, then for any G-universe $\mathcal{U}$, the transfer system $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}))$ is saturated, i.e., if $K \to H$ is admissible, and $K \leq N \leq H$ is an intermediary subgroup, then both $K \leq N$ and $N \leq H$ are admissible.*

Proof. The admissibility of $K = N \cap K \leq N$ is by restriction of $K \to H$. For the other relation, by Theorem 3.1, we need an H-embedding $\mathbb{Z}[H/N] \otimes \mathcal{U} \to \mathcal{U}$. Since $K \leq H$ is admissible, there is an H-equivariant embedding $\mathbb{Z}[H/K] \otimes \mathcal{U} \to \mathcal{U}$. It therefore suffices to find an H-embedding $\mathbb{Z}[H/N] \otimes \mathcal{U} \to \mathbb{Z}[H/K] \otimes \mathcal{U}$, or equivalently, $f : \mathbb{Z}[H] \otimes_{\mathbb{Z}[N]} \mathcal{U} \to \mathbb{Z}[H] \otimes_{\mathbb{Z}[K]} \mathcal{U}$. Write $N/K = \{n_1K, \dots, n_kK\}$. We define f as the linear extension of the assignment $h \otimes u \mapsto \sum_{i \leq k} hn_i \otimes (n_i)^{-1}u$ for any $u \in \mathcal{U}$, $h \in H$. This is well-defined since for $n \in N$, if $nn_i = n_{\sigma_n(i)}k_i(n)$ for $k_i(n) \in K$ then:

$$\begin{aligned} f(hn \otimes n^{-1}u) &= \sum_{i \leq k} hnn_i \otimes (n_i)^{-1}n^{-1}u = \sum_{i \leq k} hn_{\sigma_n(i)}k_i(n) \otimes (n_i)^{-1}n^{-1}u \\ &= \sum_{i \leq k} hn_{\sigma_n(i)} \otimes k_i(n)(nn_i)^{-1}u = \sum_{i \leq k} hn_{\sigma_n(i)} \otimes n_{\sigma_n(i)}^{-1}u \\ &= f(h \otimes u). \end{aligned}$$

This is also H-equivariant since, for all $h, h' \in H$ and $u \in \mathcal{U}$:

$$f(h \cdot (h' \otimes u)) = \sum_{i \leq k} hh'n_i \otimes (n_i)^{-1}u = h \cdot \left(\sum_{i \leq k} h'n_i \otimes (n_i)^{-1}u \right) = h \cdot f(h' \otimes u).$$

The post-composition of f by the map $\mathbb{Z}[H] \otimes_{\mathbb{Z}[K]} \mathcal{U} \to \mathbb{Z}[H] \otimes_{\mathbb{Z}[N]} \mathcal{U}$ sending $h \otimes_{\mathbb{Z}[K]} u \mapsto h \otimes_{\mathbb{Z}[N]} u$ is the map $[N : K] \cdot \mathrm{id}$, which is injective. Therefore, f is an embedding. ☐

Other necessary conditions and characterizations are proved in [Rub21b], also for the transfer systems arising from Steiner operads.

Rubin conjectured the following ([Rub21b]).

Conjecture 3.4 (Rubin's saturation conjecture). *Let $k \in \mathbb{N}^*$ and $e_1, \dots, e_k \in \mathbb{N}^*$. There exist integers $p_1, \dots, p_k$ depending on this choice such that for all k-uples of distinct primes $q_1, \dots, q_k$ with $q_i \geq p_i$ for all $i \leq k$, and $G := C_{q_1^{e_1} \cdots q_k^{e_k}}$, any saturated G-transfer system is realized by some linear isometries operad.*

The number of C_n-universes grows exponentially with $n \in \mathbb{N}$. Indeed there are $2^{\lfloor n/2 \rfloor}$ non-isomorphic C_n-universes (by Proposition 3.6 below), whereas the number of transfer systems is fixed if we fix the number of primes factors of n and their exponents, but not the primes themselves. Indeed, the subgroup poset of C_n for n with prime decomposition $p_1^{e_1} \dots p_k^{e_k}$ is isomorphic to the product poset $[e_1] \times \cdots \times [e_k]$ (with $[e_i]$ the poset $\{0 < 1 < \cdots < e_i\}$), where the subgroup $p^{f_1}C_{p^{e_1}} \times \cdots \times p^{f_k}C_{p^{e_k}} \cong C_{p^{e_1 - f_1}} \times \cdots \times C_{p^{e_k - f_k}}$ corresponds to $(e_1 - f_1, \dots, e_k - f_k)$. Therefore, when the C_n-universe varies, many linear isometries operads give rise to the same transfer system, and are therefore equivalent.

Conjecture 3.4 was proved for cyclic groups of order p^n and pq in [Rub21b], and for cyclic groups of order qp^n in [HMOO22], for $p, q \geq 5$ distinct primes and any $n \in \mathbb{N}$. In the same paper, an explicit formula for the number of saturated transfer systems on $C_{p^nq^m}$ is computed. Proposition 3.6 below, proved by Rubin, reduces the problem to an arithmetic one.

Notation. For $n \in \mathbb{N}^*$ and $0 \leq j \leq n-1$, let $\lambda_n(j)$ be the two dimensional real representation of C_n, where [1] acts by multiplication by $e^{\frac{2\pi i j}{n}}$ in the complex plane.

Definition 3.5. Let $n \in \mathbb{N}^*$. An *indexing set* for C_n is a subset $I \subseteq C_n$ such that $0 \in I$ and $-I \subseteq I$, where $-I := \{n - i \mid i \in I\}$. For each indexing set, we can define an *associated* C_n*-universe* $\mathcal{U}_I := \bigoplus_{n\in\mathbb{N}} \bigoplus_{j\in I} \lambda_n(j)$. We say that $\mathcal{U}_I$ *realizes* the associated transfer system $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}_I))$

Proposition 3.6 (5.14 and 5.15 in [Rub21b])**.** *Let $G = C_n$ for $n \in \mathbb{N}^*$.*
- *(i) Any G-universe is of the form $\mathcal{U}_I$ for some indexing set I.*
- *(ii) The relation $C_d \cong (n/d)C_n \to (n/e)C_n \cong C_e$ for $d \mid e \mid n$ is admissible in $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}_I))$ if and only if $(I \pmod e) + d = I \pmod e$ (in particular it suffices to check that $(I \pmod e) + d \subseteq I \pmod e$).*

The proofs of the special cases of the saturation conjecture mentioned above consist in building explicitly an indexing set realizing any given saturated transfer system. Applying the same method, we prove in the next section the conjecture in the case of groups of the form $C_{p^nq^m}$ with $p, q \geq 5$ distinct primes.

Remark 3.7. If either $p \leq 3$ or $q \leq 3$ (and $n, m \geq 1$), there are saturated transfer systems that are not realized by any linear isometries operad. Indeed, assume $p \leq 3$ and $n, m \geq 1$, and consider the saturated transfer system on $[n] \times [m]$ consisting of the single map $(0, 0) \to (0, 1)$ (i.e., $\{0\} \to C_q$ in terms of subgroups) and the identities. It is not realized by any linear isometries operad, by the following argument. If $I \subseteq C_{p^nq^m}$ were an indexing system realizing it, then $I \pmod{pq} := J$ realizes the restriction of the transfer system to $[1] \times [1]$, because, by Proposition 3.6, its admissible relations are characterized by translation invariance properties modulo p, q or pq, which are just the same for I. This is impossible, as proved in [Rub21b] (Lemma 5.22). Indeed, this would imply that $J \subseteq pC_{pq}$: else, $J \pmod p \neq \{0\}$, but there are only one or two (inverse to one another) non trivial element(s) modulo p if $p \leq 3$, so $J \pmod p = C_p$. Therefore, $J \pmod p$ would be invariant by translation by 1, so $\{0\} \to C_p$ would be admissible. Now, since $\{0\} \to C_q$ is admissible, $J \pmod q$ is invariant by translation by 1, so $J \pmod q = C_q$. Therefore $J = pC_{pq}$ (if $|J| < q$ then also $|J \pmod q| < q$). But then J is invariant by translation by p, and so $C_p \to C_{pq}$ is admissible (i.e., $(1, 0) \to (1, 1)$) which is a contradiction. The case $q \leq 3$ is symmetric.

Theorem 3.8 (Saturation conjecture for p^nq^m)**.** *Let $p, q \geq 5$ be distinct primes, and $n, m \in \mathbb{N}$. Let $G = C_{p^nq^m}$. Then, any saturated G-transfer system is realized by some linear isometries operad on a G-universe $\mathcal{U}$.*

Our proof in the next section is quite technical. Even if it does not bring essentially new ideas, it highlights the increasing complexity of the combinatorics one needs to understand when moving from the case p^nq to p^nq^m.

4. Proof of the main Theorem (3.8)

4.1. **Outline of the proof.** We fix $p, q \geq 5$ distinct primes, and proceed by induction on n and m. We separate base case and induction step in two lemmata.

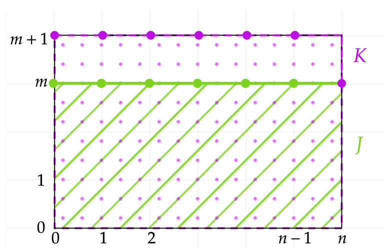
Lemma 4.1. *Let $m \geq 1$, and let $\mathcal{T}$ be a saturated $C_{q^{m+1}}$-transfer system. Given any indexing set $J \subseteq C_{q^m}$ realizing the restriction of $\mathcal{T}$ to $[m]$, there exists an indexing set $I \subseteq C_{q^{m+1}}$ realizing $\mathcal{T}$, with $I \pmod{q^m} = J$, and containing q^m.*

Any saturated $C_{q^{m+1}}$-transfer system can be realized by an indexing set containing q^i, for all $0 \leq i \leq m$.

Notation. Let $n, m \in \mathbb{N}$. We say that an indexing system $I \subseteq C_{p^n q^m}$ satisfies $(\star)$ if $I \pmod{p^{i+1} q^m}$ contains a non-zero multiple of $p^i q^m$ for all $0 \leq i \leq n-1$.

Lemma 4.2. *Let $n, m \in \mathbb{N}$. Consider a saturated $C_{p^n q^{m+1}}$-transfer system $\mathcal{T}$, and an indexing set $J \subseteq C_{p^n q^m}$ realizing the restriction of $\mathcal{T}$ to $C_{p^n q^m}$, and satisfying $(\star)$. Then, there exists an indexing set $K \subseteq C_{p^n q^{m+1}}$ realizing $\mathcal{T}$ and satisfying $(\star)$, such that K contains a non-zero multiple of $p^n q^m$ and $K \pmod{p^n q^m} = J$.*

We can imagine the situation as follows, with the large dots representing the non-zero multiples required in the different reductions of our indexing systems.



Let us now see how this implies our result. We prove the following claim by induction on m, and together with Lemma 4.1 for the case $n = 0$, this will imply Theorem 3.8.

Claim. Any saturated transfer system $\mathcal{T}$ on $C_{p^n q^m}$, with $n \geq 1$ and $m \geq 0$, can be realized by an indexing system fulfilling $(\star)$.

Proof of the Claim. Fix $n \geq 1$ an integer. We proceed by induction on m. For $m = 0$, the claim follows directly from the second part of Lemma 4.1. Assume now our claim is true for some $m \in \mathbb{N}$. Taking $\mathcal{T}$ a saturated transfer system on $C_{p^n q^{m+1}}$, by the induction hypothesis its restriction to $[n] \times [m]$ can be realized by some indexing set J satisfying $(\star)$. By applying Lemma 4.2 we get an indexing system realizing $\mathcal{T}$, and satisfying $(\star)$, as desired. □

4.2. **Proof of Lemma 4.1.** We recall once more that the saturation conjecture has already been proved by Rubin in [Rub21b] for groups of the form C_{q^m} with $m \in \mathbb{N}$ and $q \geq 5$ prime, by exhibiting an explicit indexing set. The crucial observation is that a saturated transfer system on $[m]$ is uniquely determined by its admissible cover relations: by transitivity and saturation, a relation $i \to j$ with $i < j$ is admissible if and only if all relations $k \to k+1$ with $i \leq k \leq j-1$ are admissible. We prove this case again because we need the more precise statement of Lemma 4.1 for the remainder of the proof.

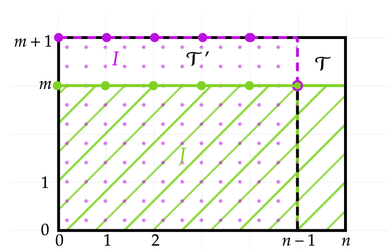
Proof. Let m, $\mathcal{T}$ and J be as in the statement. We identify J as a subset of $\{0, 1, \ldots, q^m - 1\}$. We first assume that $m \to m+1$ is admissible in $\mathcal{T}$. Let $I = \{\pm(j + \alpha q^m) \mid j \in J, 0 \leq \alpha < q\} \subseteq C_{q^{m+1}}$. Then $0 \in I$, $-I = I$, I contains q^m (since $0 \in J$), and $I \pmod{q^m} = \{\pm j \mid j \in J\} = J$. In particular, it suffices to check that I admits $m \to m+1$ to show that I realizes $\mathcal{T}$: indeed the other cover relations admissible for I match those of $\mathcal{T}$, because by Proposition 3.6 this only depends on $I \pmod{q^m} = J$ and J realizes the restriction of $\mathcal{T}$ to $[m]$. This last cover relation is admissible in I, since $I + q^m \pmod{q^{m+1}} \subseteq I$ by construction (we can always replace α by its residue modulo q since we work modulo q^{m+1}).

Now, if $m \to m+1$ is not admissible in $\mathcal{T}$, consider instead the indexing set $I = \{\pm(j + \varepsilon q^m) \mid j \in J, \varepsilon \in \{0,1\}\} \subseteq C_{q^{m+1}}$. Then as before I contains q^m, and $I \pmod{q^m} = J$. Hence, to show that I realizes $\mathcal{T}$, we only have to check that $m \to m+1$ is not admissible in I, i.e., I is not q^m-translation invariant. And indeed, $2q^m \notin I$: we have $0 < 2q^m < q^{m+1}$ since $q \geq 5$. And for all $j \in J$, $\varepsilon \in \{0,1\}$, we have $q^{m+1} - (j + \varepsilon q^m) > q^{m+1} - 2q^m = (q-2)q^m > 2q^m$ (since $q \geq 5$) and $j + \varepsilon q^m < q^m + q^m = 2q^m$. So I realizes $\mathcal{T}$, as desired.

For the second part of the statement, just begin with the trivial indexing set in C_1, and then use inductively the first part of the claim to extend it to an indexing system realizing the restriction of $\mathcal{T}$ to C_q first, and then C_{q^2} and so on. By the way we constructed these extensions above, each successive indexing set contains the previous one, in particular it contains the required powers of q. □

4.3. **Proof of Lemma 4.2.**

Proof. We take our inspiration from the proof of the case C_{pq^n} in [HMOO22]. We proceed by induction on n. The case $n = 0$ follows directly from (i) in Lemma 4.1. Assume now that the statement holds for some fixed $n-1 \in \mathbb{N}$ and every $m \in \mathbb{N}$. Let us show that it holds for n and every $m \in \mathbb{N}$. As in the statement, consider a saturated $C_{p^n q^{m+1}}$-transfer system $\mathcal{T}$, and an indexing set $J \subseteq C_{p^n q^m}$ satisfying $(\star)$ and realizing the restriction of $\mathcal{T}$ to $C_{p^n q^m}$. Let $\mathcal{T}'$ be the restriction of $\mathcal{T}$ to $[n-1] \times [m+1]$. Then the restriction of $\mathcal{T}'$ to $[n-1] \times [m]$ is realized by $J' := J \pmod{p^{n-1}q^m} \subseteq C_{p^{n-1}q^m}$, and $J' \pmod{p^{i+1}q^m} = J \pmod{p^{i+1}q^m}$ contains a non-zero multiple of $p^i q^m$ for all $0 \leq i < n-1$, so J' satisfies $(\star)$ as well. By our induction hypothesis on n, we may therefore find an indexing set $I \subseteq C_{p^{n-1}q^{m+1}}$ satisfying $(\star)$, realizing $\mathcal{T}'$, containing a non-zero multiple of $p^{n-1}q^m$, and such that $I \pmod{p^{n-1}q^m} = J'$. We illustrate the situation as follows:

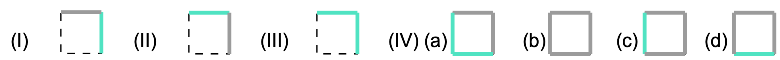


Notation. The condition $I \pmod{p^{n-1}q^m} = J \pmod{p^{n-1}q^m}$ implies that for all $i \in I$ and $j \in J$ there exists $j_i \in J, i_j \in I, \gamma_i, \delta_j \in \mathbb{Z}$ with $i = j_i + \gamma_i p^{n-1} q^m$ and $j = i_j + \delta_j p^{n-1} q^m$. Since p and q are distinct primes, by Bezout's identity, there exist $u, v \in \mathbb{Z}$ with $uq + vp = 1$. By Euclidean division, let $-u\gamma_i = r_i' p + r_i$ with $0 \leq r_i < p$ and $-v\delta_j = s_j' q + s_j$ with $0 \leq s_j < q$, for all $i \in I$ and $j \in J$.

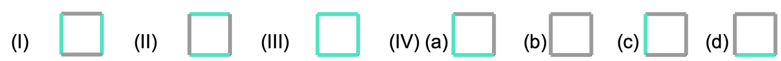
We want to find an indexing system K such that $K \pmod{p^n q^m} = J$ and $K \pmod{p^{n-1}q^{m+1}} = I$. Since I and J are fixed, and the transfer system induced by K is saturated, the latter is fully determined by the data of whether

$(n,m) \to (n,m+1)$ and $(n-1,m+1) \to (n,m+1)$ are admissible for K. Hence, K realizes $\mathcal{T}$ if and only if the admissibility of these two relations in $\mathcal{T}$ and K is the same. Indeed, all other relations except $(n-1,m) \to (n,m+1)$ are determined by I and J, and this last relation is admissible if and only if all sides of the top-right square are admissible. If K is built in this way, we have $K \pmod{p^i q^{m+1}} = I \pmod{p^i q^{m+1}}$ for all $0 \le i \le n-1$, and I satisfies ($\star$), so K satisfies ($\star$) if and only if K contains a non-zero multiple of $p^{n-1}q^{m+1}$.

Let us name the possibilities for the top-right square, with corners $(n-1,m+1)$, $(n,m+1)$, (n,m), and $(n-1,m)$:



Light blue represents admissibility in $\mathcal{T}$, and gray not being admissible. Stability under restriction, transitivity, and saturation of $\mathcal{T}$ then determine the status of the dashed maps:



Cases (I) and (III). If $(n,m) \to (n,m+1)$ is in $\mathcal{T}$, consider, in $C_{p^n q^{m+1}}$:

$$K := \pm\{r_i p^{n-1} q^{m+1} + i + kp^n q^m \mid i \in I, k \in \mathbb{Z}\} \cup (\pm\{j + kp^n q^m \mid j \in J, k \in \mathbb{Z}\}).$$

First of all, K contains $p^n q^m$ (setting $j=0$, $k=1$). To verify ($\star$), we modify our choices for j_0 and γ_0: taking $i=0$, $k=0$ in the definition of K, we have $r_0 p^{n-1} q^{m+1} \in K$ but then we want $r_0 \neq 0$ (recall that $0 \le r_0 < p$). We had $0 = j_0 + \gamma_0 p^{n-1} q^m$. By assumption J contains a non-zero multiple of $p^{n-1}q^m$, say $\alpha p^{n-1} q^m$ with $0 < \alpha < p$. Choose $j_0 = \alpha p^{n-1} q^m$ and $\gamma_0 = -\alpha$. Since $p \nmid u$, we get $p \nmid -u\gamma_0$, so $r_0 \neq 0$.

- $0 \in K$ (setting $j = 0 \in J$ and $k = 0$) and $-K \subseteq K$ by construction.
- $K \pmod{p^n q^m} = J$, since $K \pmod{p^n q^m} = \pm\{r_i p^{n-1} q^{m+1} + i \mid i \in I\} \cup (\pm J)$ (the minus signs remain unchanged because the inverse $\bmod\ p^n q^{m+1}$ of an element $0 \le x < p^n q^{m+1}$ is given by $p^n q^{m+1} - x$, but $\bmod\ p^n q^m$ this is congruent to $-x \equiv p^n q^m - x$). Moreover $\pm J = J$ because J is an indexing set. Therefore, $J \subseteq K \pmod{p^n q^m}$, and for all $i \in I$ we have
$$\begin{aligned} r_i p^{n-1} q^{m+1} + i &= -uq(\gamma_i p^{n-1} q^m) - r_i' p^n q^{m+1} + i \\ &\equiv (vp-1)(\gamma_i p^{n-1} q^m) + i = v\gamma_i p^n q^m + j_i \equiv j_i \in J \pmod{p^n q^m} \end{aligned}$$
So the other inclusion holds as well.
- $K \pmod{p^{n-1} q^{m+1}} = I$: once more we simplify
$$K \pmod{p^{n-1} q^{m+1}} = \pm\{i + kp^n q^m \mid i \in I, k \in \mathbb{Z}\} \cup (\pm\{j + kp^n q^m \mid j \in J, k \in \mathbb{Z}\}).$$
Choosing $k = 0$ we see that $I \subseteq K \pmod{p^{n-1} q^{m+1}}$. For the other inclusion, since we are in case (I) or (III), by Proposition 3.6, I is $p^{n-1}q^m$-translation invariant because I admits $(n-1,m) \to (n-1,m+1)$, so in particular it is $p^n q^m$-translation invariant. Therefore $i + kp^n q^m \in I \pmod{p^{n-1} q^{m+1}}$ $\forall k \in \mathbb{Z}, i \in I$ and
$$j + kp^n q^m = i_j + \delta_j p^{n-1} q^m + kp^n q^m = i_j + p^{n-1} q^m (\delta_j + kp) \in I.$$
Since $-I \subseteq I$, we conclude that $K \pmod{p^{n-1} q^{m+1}} \subseteq I$.

- Finally, $(n,m) \to (n,m+1)$ is in K: by Proposition 3.6, it suffices to check that K is $p^n q^m$-translation invariant, but this holds by construction.

This suffices to show that K realizes $\mathcal{T}$: indeed, both $\mathcal{T}$ and the transfer system that K realizes admits $(n,m) \to (n,m+1)$. But then, by saturation, the admissibility of $(n-1,m) \to (n,m)$ suffice for both of them to determine whether they are in situation (I) or (III), and the two transfer systems coincide there, since J realizes the restriction of $\mathcal{T}$ and the reduction of K equals J.

Case (II) (or (III)). Assume now that $\mathcal{T}$ admits $(n-1,m+1) \to (n,m+1)$. The proof is symmetric to the previous case, this time consider (in $C_{p^n q^{m+1}}$):

$$K := \pm\{i + kp^{n-1}q^{m+1} \mid i \in I, k \in \mathbb{Z}\} \cup (\pm\{s_j p^n q^m + j + kp^{n-1}q^{m+1} \mid j \in J, k \in \mathbb{Z}\})$$

Modifying s_0 as above, by using the fact that I contains a non-zero multiple of $p^{n-1}q^m$, we have that K contains a non-zero multiple of $p^n q^m$. Moreover, K contains $p^{n-1}q^{m+1}$. As above, one easily checks that $0 \in K$ and $-K \subseteq K$, and $K \pmod{p^{n-1}q^{m+1}} = I$, $K \pmod{p^n q^m} = J$ (in cases (II) and (III), J is $p^{n-1}q^m$-translation invariant (modulo $p^n q^m$), so in particular it is $p^{n-1}q^{m+1}$-translation invariant). Finally, K admits $(n-1,m+1) \to (n,m+1)$ because it is $p^{n-1}q^{m+1}$-translation invariant by construction. As above this suffices to prove that K realizes $\mathcal{T}$ because whether we are in case (II) or (III) only depends on the restriction of $\mathcal{T}$ to $[n-1] \times [m+1]$, corresponding to I, but K extends I.

Case (IV). We distinguish sub-cases:

Case (a). In this situation, I and J are both $p^{n-1}q^m$-translation invariant. In particular, they are the set of iterated translations by $p^{n-1}q^m$ of a common indexing system L in $C_{p^{n-1}q^m}$, modulo $p^{n-1}q^{m+1}$ and $p^n q^m$ respectively. Assume $q > p$ to begin with. Consider

$$K := \pm\{\ell + kp^{n-1}q^m \mid \ell \in L, 0 \le k < 2q\} \subseteq C_{p^n q^{m+1}}.$$

In particular, for $\ell = 0$ and $k = q$, K contains $p^{n-1}q^{m+1}$, respectively $p^n q^m$ for $\ell = 0$ and $k = p < 2q$, which takes care of $(\star)$. By construction K is an indexing set, and $K \pmod{p^{n-1}q^m} = \pm L = L$. Therefore it suffices to show that K is $p^{n-1}q^m$-translation invariant modulo $p^{n-1}q^{m+1}$, respectively $p^n q^m$, to show that it coincides with I, respectively J there. And indeed, for all $\ell \in L$ and $0 \le k < 2q$, we have $\ell + kp^{n-1}q^m + p^{n-1}q^m = \ell + (k+1)p^{n-1}q^m$, which is in K by definition if $k < 2q - 1$.

For $k = 2q-1$, we find $\ell + 2p^{n-1}q^{m+1} \equiv \ell \pmod{p^{n-1}q^{m+1}}$, and writing the Euclidean division $2q = sp + r$, with $1 \le r < p$ and $s \ge 1$ since $p < q$, we also have $\ell + 2qp^{n-1}q^m = \ell + sp^n q^m + rp^{n-1}q^m \equiv \ell + rp^{n-1}q^m \pmod{p^n q^m}$ which lies in K since $r < p < q$.

The argument for additive inverses is the same as the previous cases. It remains to check that K is not $p^{n-1}q^{m+1}$-translation invariant modulo $p^n q^{m+1}$. Then, since we are in case (a), by saturation, it means that it is not $p^n q^m$-translation invariant either. But $2p^{n-1}q^{m+1} < p^n q^{m+1}$ (since $p > 2$) is not contained in K: indeed, we have $0 \le \ell + kp^{n-1}q^m \le p^{n-1}q^m - 1 + (2q-1)p^{n-1}q^m = 2p^{n-1}q^{m+1} - 1$ for all

$\ell \in L$ and $0 \leq k < 2q$, and

$$\begin{aligned} p^n q^{m+1} - (\ell + kp^{n-1}q^m) &> p^n q^{m+1} - 2p^{n-1}q^{m+1} = (p-2)(p^{n-1}q^{m+1}) \\ &\geq 2p^{n-1}q^{m+1} \text{ since } p \geq 5 \end{aligned}$$

Hence K realizes $\mathcal{T}$. If $p > q$, the same proof applies by replacing $2q$ by $2p$ in the definition of K.

Case (b). Since transfer systems are closed under restriction, any indexing set extending I and J suffices. Consider

$$K := \pm\{r_i p^{n-1}q^{m+1} + i \mid i \in I\} \cup (\pm\{s_j p^n q^m + j \mid j \in J\}) \subseteq C_{p^n q^{m+1}}.$$

By modifying r_0 and s_0 as in cases (I) and (II), we obtain non-zero multiples of $p^{n-1}q^{m+1}$ and $p^n q^m$ in K. Then, as before we have

$$K \pmod{p^n q^m} = \pm\{r_i p^{n-1}q^{m+1} + i \mid i \in I\} \cup (\pm J)$$

and $r_i p^{n-1}q^{m+1} + i \equiv j_i \in J \pmod{p^n q^m}$, so $K \pmod{p^n q^m} = J$, and similarly $K \pmod{p^{n-1}q^{m+1}} = I$. Thus K realizes $\mathcal{T}$.

Case (c). We note first that any indexing system extending both I and J will not admit $(n-1, m+1) \to (n, m+1)$ by closure under restriction. It therefore suffices to find an indexing system $K \subseteq C_{p^n q^{m+1}}$, extending I and J, not admitting $(n, m) \to (n, m+1)$. Consider $\tilde{K} := \pm\{r_i p^{n-1}q^{m+1} + i \mid i \in I\} \cup (\pm J)$ and set $K := (\tilde{K} \setminus (\pm\{p^n q^m\})) \cup (\pm\{p^n q^m + \alpha u p^{n-1}q^{m+1}, 2p^n q^m\})$ where $\alpha p^{n-1}q^m \in J$ with $0 < \alpha < p$ (exists by assumption).

A non-zero multiple of $p^{n-1}q^{m+1}$, namely $r_0 p^{n-1}q^{m+1}$, is contained in $\tilde{K}$, by modifying r_0 as before, and it is contained in K too (since it cannot be equal to $\pm p^n q^m$, otherwise $p^n q^m \mid r_0 p^{n-1}q^{m+1}$ (recall that $0 \leq r_0 < p$) so $p \mid r_0 q$ but this is impossible), and K contains $2p^n q^m \not\equiv 0$ (since $q > 2$). Then $K \pmod{p^n q^m}$ clearly contains J, and $\alpha u p^{n-1}q^{m+1} = \alpha(1 - vp)p^{n-1}q^m \equiv \alpha p^{n-1}q^m \pmod{p^n q^m}$, which is contained in J by hypothesis.

Also, as before $r_i p^{n-1}q^{m+1} + i \equiv j_i \pmod{p^n q^m}$, so $K \pmod{p^n q^m} = J$. It is clear that $K \pmod{p^{n-1}q^{m+1}}$ contains I, and it is contained in I, indeed $p^n q^m + \alpha u p^{n-1}q^{m+1}, 2p^n q^m \in I$ modulo $p^{n-1}q^{m+1}$ because I is $p^{n-1}q^m$-translation invariant (case (c)); and for the same reason, for all $j \in J$, $j = i_j + \delta_j p^{n-1}q^m$ is in $I \pmod{p^{n-1}q^{m+1}}$. So $K \pmod{p^{n-1}q^{m+1}} = I$.

Finally, K is not $p^n q^m$-translation invariant. Indeed, $p^n q^m \notin K$: we have $p^n q^{m+1} - 2p^n q^m \neq p^n q^m$ since $q > 3$. Furthermore,

$$p^n q^m \not\equiv p^n q^m + \alpha u p^{n-1}q^{m+1} \pmod{p^n q^{m+1}}$$

since $p \nmid \alpha u$, and finally $p^n q^m \not\equiv -p^n q^m - \alpha u p^{n-1}q^{m+1} \pmod{p^n q^{m+1}}$ else $p^n q^{m+1}$ divides $2p^n q^m + \alpha u p^{n-1}q^{m+1}$ and p would divide αu.

Case (d). As in the previous case, it suffices to find an indexing set K extending both I and J, that does not admit $(n-1, m+1) \to (n, m+1)$. The proof is done as in the previous case, with $\tilde{K} := (\pm I) \cup \pm\{s_j p^n q^m + j \mid j \in J\}$ and

$$K := \tilde{K} \setminus (\pm\{p^{n-1}q^{m+1}\}) \cup (\pm\{p^{n-1}q^{m+1} + \beta v p^n q^m, 2p^{n-1}q^{m+1}\})$$

where $\beta p^{n-1}q^m \in I$ with $0 < \beta < q$ is our non-zero multiple of $p^{n-1}q^m$ contained in I. □

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