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The basics of $\mathbb{A}^1$-homotopy theory
and motivic Postnikov towers

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1 Introduction - Motivation for the theory

In his talk at ICM in 1998, Voevodsky defines $\mathbb{A}^1$-homotopy theory as "the homotopy theory for algebraic varieties and schemes which uses the affine line as a replacement for the unit interval" ([Voe98]). There are various reasons for the development of $\mathbb{A}^1$-homotopy theory, also called motivic homotopy theory. One of the hopes is to apply the power and advantages of topological and homotopical methods in the world of smooth schemes (over a fixed base scheme S). Notably, one goal is to produce algebraic analogs of certain topological and homotopical constructions, such as Postnikov towers, which will be one of our main focuses.

In the first place, the belief that an interesting homotopy theory should exist for schemes partially originates in several properties that give the category of schemes a topological (and homotopical) flavour. For instance, one can think about the various cohomology theories for schemes which have been defined and turned out useful. In the classical homotopy theory of topological spaces, the standard interval I plays a central role. Homotopy invariance in this context is all about "constructions agreeing on X and $X \times I$" for any topological space X. Actually, in our situation a better analogy would be a homotopy theory for smooth manifolds in which the real line plays the role of the interval. In algebraic geometry, the situation is slightly more delicate, but the affine line (with respect to our base scheme) is a good candidate to replace the interval. Some widespread constructions in algebraic geometry are $\mathbb{A}^1$-invariant, that is, they give the same result for some scheme X and for the fiber product $X \times \mathbb{A}^1$ over the base scheme. For instance, in 1976, Quillen and Suslin proved that the sets of rank r vector bundles over the spectrum of a field or over the corresponding affine n-space were in bijection. Bass and Quillen further conjectured that this result could be generalized to vector bundles over a k-scheme X and the product of X with the affine n space (for well-behaved schemes X). This was already proved in some generality. Furthermore, commonly used cohomology theories for smooth k-schemes are $\mathbb{A}^1$-invariant, such as Chow groups, Grothendieck groups and étale cohomology. It therefore seems reasonable to wish that in our homotopy theory for schemes X and $X \times \mathbb{A}^1$ should be the same up to homotopy, for any scheme X.

Two founders of $\mathbb{A}^1$-homotopy theory are Morel and Voevodsky. They introduced their $\mathbb{A}^1$-homotopy category in the article [MV99] in 1999. Originally, they were working separately on different ideas: Morel was looking into the problem of finding a natural homotopy theory for smooth algebraic varieties over a field, such that algebraic K-theory would be representable (and $\mathbb{A}^1$-invariant, more precisely such that $K_0(X)$ for a scheme X could be computed as the maps in this homotopy category between X and the classifying space of the infinite linear group BGL_∞). Voevodsky was, together with other mathematicians, looking for the category of motives, the derived category thereof, and motivic cohomology, which is supposed to be to schemes what singular cohomology is to topological spaces. The philosophy of motives was essentially introduced by Grothendieck. The idea is to unify different cohomology theories for schemes; to each scheme would be associated a motive, which encapsulates its "cohomological essence". Eventually, Morel and Voevodsky started working together and considered the idea of building a motivic homotopy theory. This collaboration started with a curious exchange of emails. Morel began by sending by the post a paper version of his "habilitation" thesis to Voevodsky, who replied with an email two months later: "Hi, are you the guy who sent me this preprint? [the title]. Best, Vladimir". Morel answered in the same style: "Yes, I am.". Then, for two weeks nothing happened, and finally Voevodsky answered back with eight PDF files, working documents, attached to his email, and they started to work together (this is the story that Morel himself told during his talk at the Vladimir Voevodsky memorial conference in München in 2018).

The construction of "the usual" homotopy theory begins with the category of topological spaces. Often, but not always, one considers instead a more convenient category of spaces, with a better behaviour (to some extent), such as the category CGWH of compactly-generated weak Hausdorff spaces. Then, one defines weak equivalences and constructs the homotopy category by localizing the category of spaces, viewed as a model category, at these maps. As a further step, one can also build the *stable* homotopy category of spectra by "stabilizing" the category of spaces with respect to the suspension function $\Sigma = S^1 \wedge -$. This process results in a spectrum being defined as a sequence of topological spaces $\{X_n\}_{n \geq 0}$, with structure maps $\Sigma X_n \to X_{n+1}$ for all $n \in \mathbb{N}$. In a certain sense, Morel and Voevodsky reproduced these steps in the motivic context, for smooth schemes of finite type over a Noetherian base scheme of finite Krull dimension. This is still much room for choices in such a construction: in the determination of a suitable category of "spaces", in the definition of weak equivalences, in the set-up for inverting the latter (are we working with homotopical categories, model categories, infinity categories...), in the choice of a suitable suspension functor to perform a stabilization, and so on.

Our main reference is the survey "A primer for unstable motivic homotopy theory" by Antieau and Elmanto ([AE17]). We do not claim originality of any the material presented here.

Now that we are (hopefully) convinced of the interest of trying to build such an $\mathbb{A}^1$-homotopy theory, let us give a rough idea of its construction. Following Antieau and Elmanto in [AE17], there are three main steps to be taken:

- Embed the category of smooth schemes into the category of simplicial presheaves on the category of smooth schemes, to ensure completeness and cocompleteness together with the existence of an interesting model structure (in our case, the projective one). The presence of simplicial sets gives a very topological flavour to the category.
- Add "*local*" or "stalkwise" equivalences to our weak equivalences. This is done via a left Bousfield localization, with respect to certain maps of simplicial presheaves that represent coverings of schemes, for a certain topology. Fibrant objects then satisfy a *hyperdescent condition*, which can be seen as a sheafification requirement.
- Impose $\mathbb{A}^1$-invariance via a second Bousfield localization, this time with respect to all projections of the form $X \times_S \mathbb{A}^1 \to X$, with X is a smooth scheme over our base scheme S and $\mathbb{A}^1$ the affine line over S.

In section 2, we undertake a somewhat detailed construction of the $\mathbb{A}^1$-homotopy category, following [AE17]. Our take will be more topological, so we assume knowledge of the basics of model categories, simplicial sets and some basic tools in homotopy theory, but do not assume advanced prerequisites in algebraic geometry. Subsection 1.2 below serves as a "crash course" introduction to the notions in algebraic geometry that we will use.

We will then get a bit of practice with motivic spaces in section 3, by doing some computations and introducing classical objects and constructions of the theory, in particular the two types of circles (the simplicial one, $\mathcal{S}^1$, and the one coming from algebraic geometry, $\mathbb{G}_m$), the *bigraded* motivic spheres, smash products and Thom spaces. All along, the parallel with topology and usual homotopy will be very present.

We will for instance compute that the suspension functor in the sense of pointed model categories is given by taking a smash product with $\mathcal{S}^1$. We will also show that some bigraded spheres admit a description in terms of smooth schemes: there are weak equivalences in the $\mathbb{A}^1$-homotopy category between $\mathcal{S}^{2n-1,n} := \mathbb{G}_m^{\wedge n} \wedge (\mathcal{S}^1)^{\wedge n-1}$ and $\mathbb{A}^n \setminus \{0\}$, and between $\mathcal{S}^{2n,n} := \mathbb{G}_m^{\wedge n} \wedge (\mathcal{S}^1)^{\wedge n}$ and the "quotient" in our category of $\mathbb{A}^n$ by $\mathbb{A}^n \setminus \{0\}$. Another computation will show that the motivic Thom space of a trivial algebraic bundle $E \to X$ in the category of smooth schemes over S is weakly equivalent to $(\mathbb{P}^1)^{\wedge n} \wedge X_+$, just as in topology, if we imagine working over the complex numbers (then the complex points in $\mathbb{P}^1$ form a 2-sphere).

In section 4, we will specialize our study to the analog in our context of homotopy groups and related notions: homotopy groups sheaves, classifying spaces, and Eilenberg-MacLane spaces.

For instance, we will obtain the usual long exact sequence in homotopy associated with a fiber sequence, and verify that our constructions have some of the properties we could legitimately expect because of their name: we will compute the homotopy groups sheaves of the candidate classifying spaces and Eilenberg-MacLane objects. We will also show that the latter represent sheaf cohomology in the usual sense.

Using these ideas, we will finally be able to provide a construction of motivic Postnikov towers in Section 5, combining the properties of simplicial sets with $\mathbb{A}^1$-homotopy. To introduce the construction, we will first discuss Postnikov towers in topology and recall a proof of their existence. Then, we will gather some facts about Postnikov-type constructions for simplicial sets. All of this will come together in the existence proof in the motivic setting. The towers will, just as in the topological case, be made of fibrations, have the "correct" $\mathbb{A}^1$-homotopy groups sheaves, and come with a notion of convergence. We will also prove that the fibrations in the towers can be chosen to be *principal twisted fibrations*, which could then be used to study a theory of obstruction in this setting, namely how to lift maps stage by stage along the Postnikov tower.

Finally, Section 6 is a miscellaneous of topics that did not fit in the other sections and that we found interesting; including diverse variations on the construction of an $\mathbb{A}^1$-homotopy theory and how they compare to the first construction we studied, the relation of $\mathbb{A}^1$-homotopy theory to the philosophy of motives, and the construction of a motivic stable homotopy category.

The Appendix 7 contains proofs of some statements about the Nisnevich topology that we will need for our discussion (the fact that it is generated by specific Nisnevich coverings with two elements, Nisnevich fibrancy of representable presheaves, and the Nisnevich descent theorem).

1.1 Conventions

- Topological spaces: the interval $[0, 1]$ will be denoted by I, and the n-sphere by $\mathbb{S}^n$ for all $n \in \mathbb{N}$.
- Algebraic geometry: for S a scheme, $\mathcal{O}_S$ denotes its structure sheaf and $\Gamma(U, \mathcal{F})$ denotes the ring of sections on an open set $U \subseteq S$ of a sheaf $\mathcal{F}$. For x a point in a scheme X, $\kappa(x)$ denotes the residue field of X at x. The affine space of dimension n over some base scheme S is denoted by $\mathbb{A}^n$ (or $\mathbb{A}^n_S$ if we want to put the emphasis on the base scheme) and are defined as the fiber products $S \times_{\mathsf{Spec}(\mathbb{Z})} \mathsf{Spec}(\mathbb{Z}[t_1, \ldots, t_n])$. In the same setting, $\mathbb{A}^n \setminus \{0\}$ or $\mathbb{A}^n \setminus S$ is the n-th affine space minus the closed subscheme S viewed as the zero section, and we define $\mathbb{G}_m := S \times_{\mathsf{Spec}(\mathbb{Z})} \mathsf{Spec}(\mathbb{Z}[t, t^{-1}]) = \mathbb{A}^1 \setminus \{0\}$ and $\mathbb{P}^1 = \mathbb{P}^1_S := S \times_{\mathsf{Spec}(\mathbb{Z})} \mathsf{Proj}(\mathbb{Z}[t])$.
- Categories and homotopy: the category of simplicial sets, respectively of *all* topological spaces, both endowed with the Quillen model structure, are denoted by $_s\mathsf{Set}$ and Top. The category Δ is the category with objects $\mathbb{N}$ that is used to construct simplicial sets as functors $\Delta^{\mathsf{op}} \to \mathsf{Set}$. We view $n \in \Delta$ as the finite set $[n] = \{0, \ldots, n\}$. Given any category C, we abuse the notation and will write $c \in C$ to signify that c is an object of C. In any category in which we have defined a notion of weak equivalence, the notation $x \simeq y$ for two objects x and y means that they are equivalent for the equivalence relation generated by weak equivalences, and $x \cong y$ means that they are isomorphic. In general, the term "essentially" will mean "up to isomorphism". For instance, a category is essentially small if the collection of isomorphism classes of objects forms a set, and up to isomorphism the collection of maps between any two fixed objects forms a set. Given $\mathcal{C}$ a model category (or more generally a homotopical category), $\mathsf{Ho}(\mathcal{C})$ denotes its homotopy category, i.e. the localization at all weak equivalences.

1.2 Notions in algebraic geometry: qcqs, flat, smooth, and étale

For definitions and statements purely in algebraic geometry, our main reference is [Vak17]. Unless otherwise mentioned, we will work over a fixed base scheme S, which we assume is quasi-compact and quasi-separated (abreviated qcqs):

Definition 1.1 (Qcqs (3.6.5, 7.3.1 and 10.1.9 in [Vak17]-)**.** A scheme S is called *quasi-compact* if it is compact as a topological space in the Zariski topology, i.e. if any open covering of S in the Zariski topology admits a finite subcover. The scheme S is called *quasi-separated* if the diagonal morphism $\Delta : S \to S \times_{\mathsf{Spec}(\mathbb{Z})} S$ is *quasi-compact*, i.e. the preimage of any affine open subset is quasi-compact.

Proposition 1.2 (Properties of qcqs schemes (5.1.H, 7.3.5 and 10.1.G in [Vak17]))**.** *Let S be a scheme.*

(i) *Qcqs lemma: If S is qcqs, the natural map $\Gamma(S, \mathcal{O}_S)_f \to \Gamma(S_f, \mathcal{O}_S)$ is an isomorphism for any section $f \in \Gamma(S, \mathcal{O}_S)$, where S_f is the non-vanishing locus of f.*

(ii) *The scheme S is qcqs if and only if it is covered by finitely many affine open sets, such that the intersection of any two of them is a finite union of affine open subsets.*

(iii) *The scheme S is quasi-separated if and only if the intersection of any two affine open sets can be covered by finitely many affine open sets.*

We work in the category Sm_S of smooth schemes over S:

Definition 1.3. We denote by Sm_S the category of smooth schemes **of finite type** over S.

Warning

From now on, every time we will talk about a smooth scheme over S, we mean an object of Sm_S, i.e. we implicitly include the finite-type assumption of the definition above.

This assumption acquires its importance in the fact that it implies the essential smallness of Sm_S (see the proof of Proposition 2.30), and this will allow us to perform the Bousfield localizations we need. It also implies that every $X \in \mathrm{Sm}_S$ is Noetherian when S is Noetherian, which will be of importance in the proof of the Nisnevich descent theorem (Theorem 2.34).

Definition 1.4 (Smooth (12.6.2 and 25.2.2 in [Vak17]))**.** A scheme X over S is called *smooth over* S if the structure morphism $X \to S$ is smooth. A morphism of schemes $f : X \to Y$ is called *smooth* if it is smooth at every $x \in X$, i.e. for every $x \in X$ there exists affine open sets $\mathrm{Spec}(A) = U \ni x$ and $\mathrm{Spec}(R) = V \subseteq Y$ with $f(U) \subseteq V$, such that the induced map $R \to A$ is a *smooth morphism of rings*: it is of finite presentation, and the *naive cotangent complex* $\mathrm{NL}_{A/R}$, namely the complex:

$$\cdots \longrightarrow 0 \longrightarrow I/I^2 \longrightarrow \underbrace{\Omega_{R[A]/R} \otimes_{R[A]} A}_{\text{degree } 0} \longrightarrow 0 \longrightarrow \dots$$

is quasi-isomorphic to a complex with a finite projective A-module in degree 0. Here, R[A] is a polynomial ring with one variable for each element of A, $I = \ker(R[A] \to A)$, and Ω denotes the module of Kähler differentials.

We say that f is *smooth of relative dimension* n for some integer $n \in \mathbb{N}$ (which implies smoothness), if any of the equivalent definitions below holds:

(i) There exists affine open covers $X = \bigcup_{i\in I} \mathrm{Spec}(B_i)$ and $Y = \bigcup_{i\in I} \mathrm{Spec}(A_i)$ such that for all $i \in I$, $f(\mathrm{Spec}(B_i)) \subseteq \mathrm{Spec}(A_i)$, and $A_i \to B_i$ is of the form $A_i \to A_i[x_1, \dots, x_{n+r}] \big/ (f_1, \dots, f_r) \cong B_i$ with the determinant of the Jacobian matrix of $(f_1, \dots, f_r)$ being a unit in B_i (this is called a *standard smooth map*).

(ii) The morphism f is locally of finite presentation, flat of relative dimension n (defined below), and the module of Kähler differentials $\Omega_{X/Y}$ is locally free of rank n.

(iii) The morphism f is locally of finite presentation and flat, and at every point $y \in Y$, the fiber X_y is a smooth $\kappa(y)$-scheme such that any of its irreducible components has dimension n.

(iv) The morphism f is locally of finite presentation and flat, and every fiber over a geometric point $\mathrm{Spec}(\overline{k}) \to Y$ (meaning that $\overline{k}$ is algebraically closed) is a smooth $\overline{k}$-scheme such any of its irreducible components has dimension n.

(v) If $Y = \mathrm{Spec}(k)$ is a field, the definition simplifies to: f is smooth of relative dimension n if every irreducible component of X has dimension n and there exists a cover of X by affine open sets of the form $\mathrm{Spec}\left(k[x_1, \dots, x_n] \big/ (f_1, \dots, f_r)\right)$ such that the Jacobian matrix of $(f_1, \dots, f_r)$ has corank (dimension of the cokernel) n at all points. Then this holds for *all* affine covers of this form.

In particular, a smooth morphism is already locally of finite presentation (and therefore locally of finite type). In Sm_S we further ask our scheme to be of finite type over S.

A flat morphism is defined as follows:

Definition 1.5 (Flat (24.2.6 and 24.5.7 in [Vak17]))**.** A morphism $f : X \to Y$ is *flat* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module. If moreover f is locally of finite type such that all fibers have all their irreducible components of dimension n, f is called *flat of relative dimension* n.

Definition 1.6 (Etale (12.6.2 and 25.2.C in [Vak17]))**.** A morphism of schemes $f : X \to Y$ is called *étale* if any of the equivalent definitions below holds:

(i) For every $x \in X$, there exists affine open sets $\mathsf{Spec}(A) = U \ni x$ and $\mathsf{Spec}(R) = V \subseteq Y$ with $f(U) \subseteq V$, such that the induced map $R \to A$ is a *étale morphism of rings*: it is of finite presentation, and the naive cotangent complex $NL_{A/R}$ (see in Definition 1.4 above) is quasi-isomorphic to zero.

(ii) For every $x \in X$ there exists affine open sets $\mathsf{Spec}(A) = U \ni x$ and $\mathsf{Spec}(R) = V \subseteq Y$ with $f(U) \subseteq V$, such that the induced map $R \to A$ is a smooth morphism of rings and $\Omega_{A/R} = 0$.

(iii) There exists affine open covers $X = \bigcup_{i \in I} \mathsf{Spec}(B_i)$ and $Y = \bigcup_{i \in I} \mathsf{Spec}(A_i)$ such that for all $i \in I$, $f(\mathsf{Spec}(B_i)) \subseteq \mathsf{Spec}(A_i)$, and the induced map $A_i \to B_i$ is of the form $A_i \to A_i[x]_h / (g) \cong B_i$ with $h, g \in A_i[x]$, g monic and $\frac{\partial g}{\partial x}$ a unit in B_i (this is called a *standard étale map*).

(iv) The morphism f is smooth and unramified.

(v) The morphism f is smooth of relative dimension 0.

(vi) The morphism f is locally of finite presentation, flat and unramified.

(vii) The morphism f is locally of finite presentation, flat and for every $y \in Y$, the fiber X_y is the spectrum of a product of finite separable extensions of $\kappa(y)$ (an *étale cover of* $\kappa(y)$).

(viii) The morphism f is locally of finite presentation, flat and for every geometric point y represented by $\mathsf{Spec}(\overline{k}) \to Y$, the corresponding fiber is the disjoint union of copies of $\mathsf{Spec}(\overline{k})$.

(ix) If $Y = \mathsf{Spec}(k)$ is a field, the definition simplifies to: X has an affine open cover $\bigcup_{i \in I} \mathsf{Spec}(A_i)$ such that $\mathsf{Spec}(A_i) \to \mathsf{Spec}(k)$ expresses A as a finite product of finite separable field extensions of k.

Proposition 1.7 (Properties of flat, smooth and étale maps (12.6.3, 12.6.C, 24.2.E, 24.5.G in [Vak17]).

(i) All three classes of morphisms are stable under composition, base change, and contain open immersions.

(ii) A morphism of locally Noetherian schemes which is either flat and locally of finite presentation, or smooth, or étale, is open.

Let us try to get some intuition about these definitions, following [Vak17].

Smoothness for a scheme is usually the strongest "non-singularity notion" one can ask for. Another "non-singularity notion" is the concept of a regular scheme. This asserts that the dimension of the Zariski (co)tangent plane should be equal to the dimension of the scheme at that point, ruling out singularities of the following kinds, where there are "too many tangent directions":

Figure 1: Some non-regular schemes (figure taken from [Vak17], Section 12.3)

For affine varieties over an algebraically closed field, in some cases regularity can be checked using the Jacobian criterion, which is very reminiscent of usual differential geometry and the implicit function theorem. But regularity is an *intrinsic* notion for a scheme. On the other hand, in algebraic geometry, many good notions are actually *relative* to a base scheme: a scheme X over S has a certain property (over S) if the structure morphism $X \to S$ has this property. For example, separatedness or properness are defined relatively to the base scheme (eventually $\mathsf{Spec}(\mathbb{Z})$ if no base scheme is specified). The same is true for smoothness. Over any field, smoothness implies regularity. And actually, for varieties over a perfect field, regularity and smoothness are the same notion. But in a more general context, this fails.

Example 1.8 (Regular non-smooth scheme, 12.2.11 in [Vak17]). Let p be an odd prime and $k = \mathbb{F}_p(u)$. Then for $X = \mathsf{Spec}\left(k[x,y] / (y^2 - x^p + u) \right)$, the closed point $(y, x^p - u)$ is regular but non-smooth.

The notion of a smooth morphism is the algebraic analog of submersions for manifolds, which induce surjections on the tangent spaces, whereas étale morphisms play the role of covering spaces, which induce bijections on the tangent spaces (although we will see that this intuition is not totally exact). In the same spirit, unramified morphisms would correspond to immersions, which induce injections on the tangent spaces. The following picture represents an étale map, more precisely a degree 2 "covering":

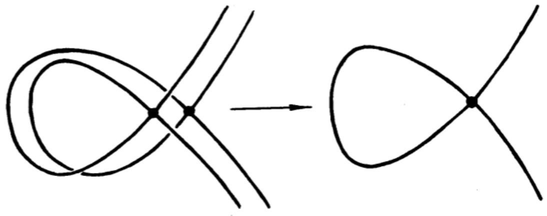


Figure 2: An étale covering of the nodal curve (figure taken from [Har77], Exercise III.10.6.)

One can also view etale morphisms as maps to an open subscheme that satisfy an algebraic version of the implicit function theorem (the latter does not hold in general for schemes) (see 29.2.F and 12.6.A in [Vak17]). There are also "functorial" definitions of these notions, for instance the notion of *formal smoothness* (see 25.5.2 in [Vak17]).

Example 1.9 (Étale and non-étale morphisms). Here are some additional examples of étale and non-étale maps. Let k be a field, and $\mathbb{A}^1_k = \mathbb{A}^1$ the affine line.

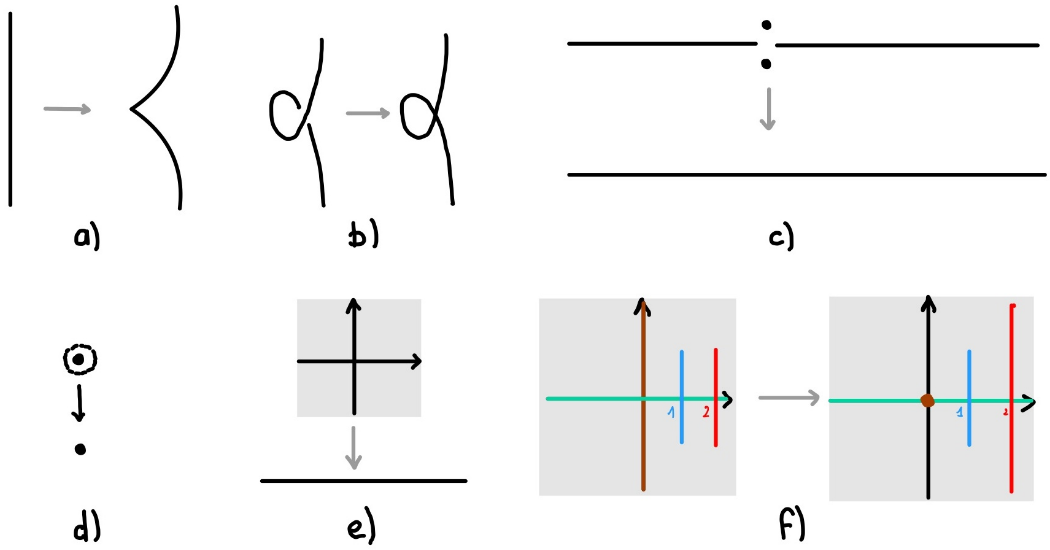


Figure 3: Various morphisms of schemes

The morphism in a) is the normalisation of the cusp: $\mathsf{Spec}(k[t]) \to \mathsf{Spec}\left(k[x,y]/(y^2-x^3)\right)$ represented by the ring map $x \mapsto t^2$ and $y \mapsto t^3$. This morphism is not smooth (and so not étale either) because it is ramified at the origin: the fiber over (x,y) is the spectrum of $k \otimes_{k[x,y]/(y^2-x^3)} k[t]$ where both x and y are sent to 0 in the left-hand side. This is isomorphic to $k[t]/(t^2)$ (a non-reduced point) over k, so it is ramified.

The morphism in b) is the blow-up of the nodal curve: $\mathbb{A}^1 \to \mathsf{Spec}\left(k[x,y]/(y^2-x^2-x^3)\right)$, represented by the ring homomorphism $x \mapsto t^2-1$ and $y \mapsto t^3-t$. This is non ramified, but it is not smooth since it is not flat. Indeed, $\mathcal{O}_{\mathbb{A}^1,(t-1)}$ is not a flat as a module over $\left(k[x,y]/(y^2-x^2-x^3)\right)_{(x,y)}$. To see this, we use the following algebraic fact (exercise 1.2.10 in [Liu02]):

For A an integral domain, its integral closure B is flat and finitely generated as an A-module if and only if $A = B$ is integrally closed.

We apply this statement to $A = \left(k[x,y](y^2 - x^2 - x^3)\right)_{(x,y)}$ and $B = \mathcal{O}_{\mathbb{A}^1,(t-1)}$ (it is the integral closure of A since the morphism we are considering is a normalization morphism). Then B cannot be flat over A, otherwise A would be integrally closed and the cusp would already be a normal curve.

The morphism in $c)$ in the projection from the line with two origins, obtained by gluing two copies of $\mathbb{A}^1$ along $A^1 \setminus 0 \cong \mathsf{Spec}(k[t,t^{-1}])$, onto the affine line. This is an étale map, because this property is local on the source, and on each copy of the affine line in the source space, our morphism restricts to the identity. Hence étale maps represent a notion of covering, but do not exactly correspond to covering spaces in topology, because otherwise all fibers (on a connected component) should have the same cardinality.

The morphism in $d)$ is $\mathsf{Spec}\left(k[t]/(t^2)\right) \to \mathsf{Spec}(k)$, from a non-reduced point to a reduced one. It is ramified and hence non étale. It is non smooth either, an indirect way to see this is to note that if was smooth, its relative dimension would be zero because we have two zero-dimensional schemes. It would then be étale. Other variations of "point(s) to point" morphisms include for example $\mathsf{Spec}\left(k[t]/(t^2-4)\right) \to \mathsf{Spec}(k)$. This time, in characteristic different from 2, the source scheme is isomorphic to $\mathsf{Spec}(k) \amalg \mathsf{Spec}(k)$ because the ring decomposes as a product by the Chinese remainder theorem. On each point, the map restricts to the identity. Hence it is étale.

The morphism in $e)$ is the projection $\mathbb{A}^2_k \to \mathbb{A}^1$ from the affine plane to the affine line. It is smooth but non étale: indeed it is an standard smooth map of affine schemes, but each fiber is one-dimensional, so the relative dimension is not 0 and the morphism cannot be étale.

Finally, the morphism in $f)$ is given by $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$, induced by $x \mapsto x$, $y \mapsto xy$. It is not étale, indeed the fiber above (x,y) is one dimensional. It is not smooth either, because smooth morphisms are flat and another feature of flat morphisms is that the fibers should be of the "expected dimension", i.e. the difference of the (local) dimensions of the target and the source. More formally, by Proposition 24.5.6 in [Vak17], we would have in our case:

$$\underbrace{\mathsf{codim}_{\mathbb{A}^2_k}(x,y)}_{2} = \underbrace{\mathsf{codim}_{\mathbb{A}^2_k} f((x,y))}_{2} + \underbrace{\mathsf{codim}_{f^{-1}((x,y))}(x,y)}_{1}$$

which is absurd (fiber at (x,y) has dimension 1 instead of the expected dimension 0).

2 Construction of the $\mathbb{A}^1$-homotopy category

The construction that will be presented in this section can be made more general, using the approach of Morel and Voevodsky in [MV99], for any "site with an interval" (see Section 2.3 p 85 in [MV99]). Here $\mathbb{A}^1$ will be our interval object. There are several variations in the literature in the construction of a motivic category. We will follow the approach presented in [AE17], which is the same as the one in [DRØ03] for instance, but differs slightly from the construction in [MV99]. We will discuss these differences in Section 6.

To motivate our discussion, let us mention that if one applies this construction to the category of smooth manifolds (with the suitable topology) and the real line as an interval object, then one obtains a homotopy theory equivalent to that of topological spaces (see [Dug01b], Theorem 8.3 for instance, or the incomplete draft [Dug98]).

2.1 A convenient category of spaces

When trying to build a homotopy theory for schemes, the first problem that one runs into is the absence of limits and colimits in general. Although constructions like fiber products always exist, in general the situation with respect to limits and colimits is quite bad.

Example 2.1 (Schemes do not admit all colimits). We will show that the category Sm_S does not admits all pushouts when $S = \mathsf{Spec}(A)$ an affine integral scheme (if S is not affine but still integral, we may just consider a smaller affine open subset in S, and then schemes over this smaller affine subset are also schemes over S via the inclusion; this preserves smoothness). This example is inspired by the MathOverflow post [hga]. Note that the affine line $\mathbb{A}^1 = \mathsf{Spec}(A[t])$ is a smooth (integral) S-scheme (the map $A \to A[t]$ is a standard smooth map of rings). We claim that the gluing of two copies of $\mathbb{A}^1$ by their generic point does not exist in Sm_S: assume for a contradiction that there exists a pushout diagram in Sm_S (or just in the category of schemes):

$$\begin{array}{ccc} \{\eta\} = \mathsf{Spec}(\mathsf{Frac}(S)(t)) & \to & \mathbb{A}^1 \\ \downarrow & & \downarrow g \\ \mathbb{A}^1 & \xrightarrow{f} & X \end{array}$$

Then, there is a natural map $\mathbb{A}^1 \amalg \mathbb{A}^1 \to X$ induced by f and g, and it is injective on closed points: indeed, if x is a closed point in the first copy of $\mathbb{A}^1$ and y in the second copy, then consider the scheme X' equal to the affine line with the point x doubled, formally constructed as the gluing of two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{x\}$ (now this gluing exists in the category of all schemes because $\mathbb{A}^1 \setminus \{x\}$ is an open subscheme of the affine line). This is smooth over S because locally, $X' \to S$ is just the morphism $\mathbb{A}^1 \to S$ (point c) in Example 1.9). There is a map $X \to X'$ induced by the universal property of the pushout, sending the first copy of $\mathbb{A}^1$ into the first copy of $\mathbb{A}^1$ in X', respectively the other copy into the other copy. The generic point η of $\mathbb{A}^1$ is sent to the same image by these two maps, because they coincide on the open subset $\mathbb{A}^1 \setminus \{x\}$, which contains η because any open subset contains the generic point. But now, if $x \neq y$, clearly they have different images by the composition $\mathbb{A}^1 \amalg \mathbb{A}^1 \to X \to X'$, and if $x = y$, each one of them is sent to one of the copies of x in X', so their images under $\mathbb{A}^1 \amalg \mathbb{A}^1 \to X \to X'$ are again different.

But now if $U \ni f(\eta) = g(\eta)$ is an affine open subset in X, then $f^{-1}(U)$ and $g^{-1}(U)$ are dense open subsets of $\mathbb{A}^1$ (they contain the generic point by construction). Let $V = f^{-1}(U) \cap g^{-1}(U)$, viewed as an open subscheme of $\mathbb{A}^1$. Then f and g induce two maps $V \to U$, that send the generic point to the same image. These are morphisms from an integral scheme (in particular, reduced) to a separated scheme (since it is affine), which agree on a dense subset, so they agree everywhere. Now V being dense in $\mathbb{A}^1$, it contains a closed point, contradicting the fact that f and g do not send any closed point of $\mathbb{A}^1$ to the same image.

The choice of the category of *finite type smooth* schemes has advantages in terms of the size of the categories involved, indeed we will see later that Sm_S is essentially small. Smoothness, together with the other choices made later in the construction, also allow the use of "localization" techniques, to reduce questions to the case where the base scheme is the spectrum of a field. Over a field, additional interesting tools are at disposal. For more details, see the MathOverflow post [hh].

2.2 An appropriate topology

2.2.1 Presheaves

One way of adding (co)limits to an essentially small category is to embed it into a category of functors with values in some (co)complete category. We also want to have a model structure on the resulting category (that is sufficiently interesting), and to be able to perform topological constructions. So the choice of the category ${}_{\mathsf{S}}\mathsf{Set}$ of simplicial sets (which is a model for topological spaces) seems adapted. By choosing the category of sets for example, we would not obtain any kind of interesting homotopical information. Therefore, we define:

Definition 2.2 ((Simplicial) presheaves)**.** For $\mathcal{C}$ an essentially small category and $\mathcal{D}$ another category, the category of *$\mathcal{D}$-valued presheaves on $\mathcal{C}$* is defined as $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathcal{D})$. In the case $\mathcal{D} = \mathsf{Set}$, respectively $\mathcal{D} = {}_{\mathsf{S}}\mathsf{Set}$, one simply talks about *presheaves on $\mathcal{C}$*, respectively *simplicial presheaves on $\mathcal{C}$*, and the corresponding category is denoted by $\mathsf{Pre}(\mathcal{C})$, respectively ${}_{\mathsf{S}}\mathsf{Pre}(\mathcal{C})$.

Any simplicial presheaf can also be seen as a simplicial object in the category of presheaves on $\mathcal{C}$, or even as a presheaf on $\Delta \times \mathcal{C}$. We will use this change of perspective several times.

Since categories of functors to a complete and cocomplete category have the same property, we have obtained a complete and cocomplete category ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ of simplicial presheaves on Sm_S. We consider the Quillen model structure on ${}_{\mathsf{S}}\mathsf{Set}$. As a category of ${}_{\mathsf{S}}\mathsf{Set}$-valued functors, the category ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ admits a projective model structure, making it into a simplicial model category (see [BK72] p 314). The weak equivalences, respectively fibrations, are defined objectwise, and cofibrations are defined by the suitable left lifting property. The model category obtained is also combinatorial, by Proposition A.2.8.2 in [Lur09].

We have a sequence of embeddings:

$$\mathsf{Sm}_S \longrightarrow \mathsf{Pre}(\mathsf{Sm}_S) \longrightarrow {}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$$

where a smooth scheme X over S is sent to the representable presheaf $\mathsf{Sm}_S(-, X)$ (this is the Yoneda embedding), and then to the constant simplicial object corresponding to it. We call the objects in the image of this embedding "representable presheaves" or "representables objects" when the context is clear enough.

This finishes the first step of the construction of a "motivic category". We will also need a pointed version of our constructions: let ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)_*$ be the category of pointed simplicial presheaves and pointed morphisms. A basepoint for $X \in {}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ is just the choice of a map $* \to X$ where $*$ is the constant trivial simplicial presheaf, or equivalently the image of S by the embedding described above. Pointed simplicial presheaves are the same as presheaves of pointed simplicial sets, or presheaves of simplicial pointed sets. Representable objects in ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ do not come with a canonical choice of a basepoint (although for some schemes like $\mathbb{A}^1$ we will later choose a particular basepoint); but one can add a disjoint basepoint to the presheaf they represent. We denote this construction for a representable object U by $U_+ := U \amalg *$. This can also be applied to objects that are not representable.

The model structure on the pointed version of the category is the same: a pointed map is a fibration/cofibration/weak equivalence if and only if the forgetful functor sends it to a map of the same type in the non-pointed version of the category.

We will highlight the main steps of the construction like this:

Step 1

The simplicial model category ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ of simplicial presheaves of smooth schemes with the projective model structure, where a scheme $X \in \mathsf{Sm}_S$ embeds as the representable functor $\mathsf{Sm}_S(-, X)$, viewed as a constant simplicial object.

(Respectively, the simplicial model category ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)_*$ of pointed simplicial presheaves.)

This model category is also a monoidal model category with respect to the cartesian monoidal structure (see for example the page "Model structure on simplicial presheaves" in the nLab, Lemma 8.1).

2.2.2 Sheaves and sites

For the definitions and results in this section, our reference is the Stacks project, Tag 00UZ.

As in algebraic geometry, together with the notion of presheaf on a scheme (or a topological space) comes the notion of a sheaf (for the Zariski topology). Here we are considering presheaves on categories, so the corresponding notion of a sheaf will require a notion of topology on a category:

Definition 2.3 (Grothendieck topology)**.** Let $\mathcal{C}$ be a category with finite limits (at least, pullbacks). A *Grothendieck topology* on $\mathcal{C}$ is the data of a collection τ of families of maps $\{f_i : U_i \to X\}_{i\in I}$ (for each family, the sources may vary but not the target), called *coverings*, satisfying the following axioms:

- *Base change* of coverings are coverings: if $\{f_i : U_i \to X\}_{i\in I}$ is a covering and $Y \to X$ a morphism in $\mathcal{C}$, then $\{U_i \times_X Y \to Y\}_{i\in I}$ is a covering.
- *Refinement*: let $\{g_j : V_j \to X\}_{j\in J}$ be any family of maps and $\{f_i : U_i \to X\}_{i\in I}$ be a covering. If for all $i \in I$, $\{V_j \times_X U_i \to U_i\}_{j\in J}$ is a covering, then $\{g_j : V_j \to X\}_{j\in J}$ is a covering.
- *Sections*: if $\{f_i : U_i \to X\}_{i\in I}$ is a family of maps such that some f_i admits a section $s : X \to U_i$, i.e. $f_i \circ s = \mathrm{id}_X$, then this family is a covering.

We say that the category $\mathcal{C}$ together with this collection τ of covering forms a *(Grothendieck) site* $(\mathcal{C}, \tau)$.

We will give some more motivation for this definition at the beginning of subsection 2.2.3.

Example 2.4 (Sites).

- The most basic example of a site is that of a topological space X. Consider the poset category $\mathsf{Open}(X)$ of open sets in X ordered by inclusion. We say that a family $\{f_i : U_i \to U\}_{i\in I}$ is a covering if and only if $\bigcup_{i\in I} U_i = U$. This defines a Grothendieck topology τ, and the site $(\mathsf{Open}(X), \tau)$ is called the *small topological site over* X.
- The *big topological site over* X consists is the comma category $\mathsf{Top} \downarrow X$ with covering the families of open immersions $\{f_i : U_i \to U\}$ over X, such that $\bigcup_{i\in I} f_i(U_i) = U$.
- More generally, for X a scheme, we can define a whole family of sites as follows: let E be a class of morphisms of schemes that contains the isomorphisms, and is stable by composition and base change (for instance, smooth maps, or étale maps, or quasi-separated maps, and so on). The *small* E*-site over* X consists in the category of schemes over X such that the structure morphism is in E, together with the topology generated by E-coverings: families of morphisms over X that are in E, and such that the union of their images cover the target scheme.
- The *big* E*-site over* X consists instead of the whole category $\mathsf{Sch} \downarrow X$ (the structure morphism is not required to be in E anymore), with coverings defined in the same way as for the small E-site. In particular, the *Zariski site* is the big E-site with E the class of open immersions. There are some subtleties here, we cannot just consider plain E-coverings because then the “Sections” axiom in Definition 2.3 would not be satisfied: a family consisting of a single morphism not in E but admitting a section should still form a cover. If we try to fix this by adding arbitrary families such that one map admits a section to our coverings, this creates difficulties with the “Refinement of coverings” axiom, and so on. To fix this problem we have to consider the topology *generated* by E-coverings instead.

Here is one way of expressing generating data for a Grothendieck topology:

Definition 2.5 (cd-structure)**.** A *cd-structure* on a category $\mathcal{C}$ with finite limits is a class of commutative squares in $\mathcal{C}$ that is stable under isomorphisms.

The acronym “cd” stands for “completely decomposable”. Any cd-structure gives rise to a Grothendieck topology on $\mathcal{C}$, by considering the coarsest topology (the one with the least coverings) such that for any commutative square

$$\begin{array}{ccc} a & \to & b \\ \downarrow & & \downarrow \\ c & \to & d \end{array}$$

in the cd-structure, $\{b \to d, c \to d\}$ is a covering.

One can also define continuous functors, i.e. morphisms of sites, to make the analogy with topological spaces more complete. However, the definition is a bit intricate and we will not need it in our discussion.

We are now in position to define sheaves:

Definition 2.6 (Sheaves on a site)**.** Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{D}$ a category with all products. Then a *sheaf* on this site (also called a τ-sheaf) is a $\mathcal{D}$-valued presheaf on $\mathcal{C}$ such that for any $\{f_i : U_i \to U\}_{i \in I} \in \tau$, the following sequence is an equalizer in $\mathcal{D}$:

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)$$

and the first map is a monomorphism.

There is a functorial way to turn presheaves into sheaves:

Proposition 2.7 (Sheafification (see the Stacks project, Tag 00ZG))**.** *Let $\mathcal{C}$ be a site. Then the inclusion of the category of presheaves of sets on $\mathcal{C}$ into the category of sheaves on $\mathcal{C}$ admits a left adjoint, called the* sheafification functor*. This functor is exact (it preserves finite limits and finite colimits).*

Remark 2.8. When dealing with simplicial presheaves we will simply perform a levelwise sheafification; by functoriality of the construction the result is still a simplicial presheaf. Moreover, it is a sheaf because the limits of simplicial sets that will appear when applying the definition of a sheaf are also computed levelwise.

Example 2.9. Any (locally small) category $\mathcal{C}$ admits a canonical Grothendieck topology: the largest *subcanonical topology*, i.e. the largest one such that all representable presheaves are sheaves with respect to this topology. Alternatively, it can be described as the topology whose coverings are the *universal effective epimorphisms*, namely families $\{f_i : U_i \to U\}_{i \in I}$ such that

$$\mathcal{C}(U, Z) \longrightarrow \prod_{i\in I} \mathcal{C}(U_i, Z) \rightrightarrows \prod_{i,j\in I} \mathcal{C}(U_i \times_U U_j, Z)$$

is an equalizer of sets for all $Z \in \mathcal{C}$ and the same holds for all base changes of this family.

Sheaves are basically presheaves that are of a local nature. The equalizer in Definition 2.6 represents the fact that sheaves are determined by their values on the coverings for the topology τ. For example, a presheaf $\mathcal{F}$ on $\mathcal{C}$ valued in a concrete category is a sheaf if and only for any object $X \in \mathcal{C}$ and τ-covering $\{U_i \to X\}$, for any choice of sections $s_i \in \mathcal{F}(U_i)$, such that the restrictions of s_i and s_j to $U_i \times_X U_j$ (their images in $\mathcal{F}(U_i \times_X U_j)$) agree, they "glue" into a unique section in $\mathcal{F}(X)$ such that its restriction to U_i is s_i for all i.

Now the question before us is to know what topology we should use on the category ${}_S\mathrm{Pre}(\mathrm{Sm}_S)$. Actually, we will instead choose an appropriate topology on Sm_S, and then consider a particular class of maps in ${}_S\mathrm{Pre}(\mathrm{Sm}_S)$ that *represent coverings in* Sm_S, in a sense yet to be precised.

2.2.3 Topologies on the category of smooth schemes

The category of smooth schemes has many different commonly used Grothendieck topologies. To name only a few, there are the Zariski, étale, Nisnevich, or cdh topologies.

The *Zariski topology* is simply the small or big E-site for E the class of open immersions, as introduced in 2.4. The *étale topology* is the small or big E-site where E is the class of étale morphisms. The *cdh topology* is the one with coverings generated by Nisnevich coverings (which we will define just below) and families of the form $\{Y \to X, Z \to X\}$ where $Y \to X$ is a finitely presented proper morphism, $Z \to X$ is a finitely presented closed immersion, and $Y \to X$ induces an isomorphism on $X \setminus Z$.

The Zariski topology on an individual scheme X is too coarse for many purposes: for instance, an irreducible scheme with the Zariski topology is contractible as a topological space (it deformation-retracts to its generic point). However, it is not easy to add open sets to it in a meaningful way. This is why one considers various Grothendieck topologies on the category of schemes instead. From a slightly different perspective, one can view Zariski open sets in X as schemes over X, represented by an open immersion $U \to X$. Now, to "add open sets", we think of an open set as being a scheme over X, that comes with a structure morphism belonging to some fixed class of (open) morphisms, more general than open immersions, for instance étale maps. These new "open sets" come with the information of how they "lie over X". Grothendieck topologies formalize this point of view at the global level of the category of schemes.

The construction of the $\mathbb{A}^1$-homotopy theory in [AE17] uses Nisnevich topology.

Definition 2.10 (Nisnevich topology)**.** The *Nisnevich topology on* Sm_S is defined as the Grothendieck topology generated by *Nisnevich coverings*: these are the finite families of étale morphisms (see Definition 1.6) $\{p_i : U_i \to X\}_{i=1}^r$ such that there exists a finite filtration $\emptyset = Z_n \subsetneq \cdots \subsetneq Z_0 = X$ of X by finitely presented closed subschemes, such that the maps

$$\coprod_{i \le r} p_i^{-1}(Z_m \setminus Z_{m+1}) \longrightarrow Z_m \setminus Z_{m+1}$$

admit a section for all $0 \le m \le n-1$.

Equivalently (see [Hoy16]), the condition about the filtration can be replaced by the requirement that the maps $\{p_i\}_{i=1}^r$ should be jointly surjective on the k-points of X for any field k. If S is Noetherian of finite Krull dimension, this is equivalent to the requirement that for all $x \in X$, there exists $i \le r$ and $\bar{x} \in U_i$ such that $p_i(\bar{x}) = x$, and p_i induces an isomorphism of residue fields $\kappa(x) \to \kappa(\bar{x})$.

For us, the terms "Nisnevich covering" will really designate these generating coverings, and the other ones will simply be called "coverings with respect to the Nisnevich topology". However, if S is Noetherian, schemes in Sm_S are Noetherian too because they are smooth and of finite type over S. In particular there are quasi-compact and thus any covering with respect to the Nisnevich topology will contain a Nisnevich covering. Therefore, in this situation, making this distinction is not very relevant.

The Nisnevich topology is finer than the Zariski topology, but coarser than the étale topology:

- every (finite) Zariski covering is a Nisnevich covering: let $\{p_i : U_i \hookrightarrow X\}_{i=1}^r$ be a finite Zariski covering. Consider the filtration of closed subschemes $Z_i := X \setminus \left(\bigcup_{j=1}^i U_i\right)$ so that $Z_r = \emptyset$ and $Z_0 = X$. Then, for all $0 \le m \le r-1$, we have:

$$\begin{aligned}\coprod_{i \le r} p_i^{-1}(Z_m \setminus Z_{m+1}) = \coprod_{i \le r}(U_i \cap (Z_m \setminus Z_{m+1})) &= \coprod_{i \le r}\left(U_i \cap \left(U_{m+1} \setminus \bigcup_{j \le m} U_j\right)\right) \\ &= \left(U_{m+1} \setminus \bigcup_{j \le m} U_j\right) \amalg \left(\coprod_{i=m+2}^{r}\left(U_i \cap \left(U_{m+1} \setminus \bigcup_{j \le m} U_j\right)\right)\right) \\ &\longrightarrow U_{m+1} \setminus \bigcup_{j \le m} U_j = Z_m \setminus Z_{m+1}\end{aligned}$$

  clearly admits a section given by inclusion in the first component.
- every Nisnevich covering is an étale covering: by definition, all maps in a Nisnevich covering $\{p_i : U_i \to X\}_{i=1}^r$ are étale, so it suffices to show that the union of their image equals the target scheme. For any $x \in X$, pick the largest integer m such that $x \in Z_m$ in the filtration $\{Z_i\}_{i=0}^n$ of X in our definition (in particular $m < n$). Then $x \in Z_m \setminus Z_{m+1}$ and the map $\coprod_{i \le r} p_i^{-1}(Z_m \setminus Z_{m+1}) \to Z_m \setminus Z_{m+1}$ is surjective because by assumption it admits a section. So x lies in the image of p_i for some $i \le r$, as desired.

Zariski coverings excepted, the simplest examples of Nisnevich coverings are the following ones:

Definition 2.11 (Elementary Nisnevich square)**.** A commutative square in Sm_S is called an *elementary (distinguished) Nisnevich square* if it is a pullback square of the form:

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & \lrcorner & \Big\downarrow {\scriptstyle \text{étale}}\; p \\ U & \underset{\text{open immersion}}{\xrightarrow{\ \iota\ }} & X \end{array}$$

with p inducing an isomorphism $p^{-1}(X \setminus U) \to X \setminus U$.

Indeed, if we have an elementary Nisnevich square as above, then $\{U \to X, V \to X\}$ is a Nisnevich covering: open immersions are étale by proposition 1.7, and $Z_2 = \emptyset$, $Z_1 = X \setminus U$, $Z_0 = X$ defines a filtration. Then $\iota^{-1}(Z_1) \amalg p^{-1}(Z_1) = p^{-1}(Z_1) \to Z_1 = X \setminus U$ has a section, because it is an isomorphism by hypothesis, and $\iota^{-1}(Z_0 \setminus Z_1) \amalg p^{-1}(Z_0 \setminus Z_1) = U \amalg p^{-1}(U) \to U$ clearly has a section given by inclusion in the first component.

Proposition 2.12 (cd-structure for Nisnevich topology)**.** *The elementary Nisnevich squares form a cd-structure that generates the Nisnevich topology of Definition 2.10.*

Proof. The proof can be found in the Appendix, in Subsection A.1, where elementary Nisnevich squares are studied in more details. □

The choice of the Nisnevich topology Since étale morphisms are open, the images of Nisnevich coverings are just Zariski open sets and do not define any new interesting topology on a fixed scheme (really as a topology on a set and not a Grothendieck topology). So the Nisnevich topology, in comparison with the Zariski topology, has more information and lives at the level of the category of schemes, and not on each individual scheme. In some sense, this is good because the Zariski topology is notoriously too coarse for many purposes. Nisnevich topology has many properties that make it desirable for our construction (see the MathOverflow post [hpl]). Firstly, it is generated by a cd-structure as we saw. This is also the case of the Zariski topology, but not for the étale one (claimed in [MV99], p 96, before proposition 1.4 in section 3). Also, the Nisnevich topology is *subcanonical*:

Lemma 2.13 (Nisnevich topology is subcanonical)**.** *Let τ_{Nis} be the Nisnevich topology on Sm_S. Every representable presheaf on the site (Sm_S, τ_{Nis}) is a sheaf.*

Proof. Since every Nisnevich covering is an étale covering (see below Definition 2.10), the result follows from the fact that étale topology is subcanonical. This well-known property is proved for example in the Stacks project, Tag 03NV. □

There are many other much deeper justifications for the choice of this topology. In particular, Antieau and Elmanto ([AE17], p 3) present it as the coarsest topology such that the purity theorem holds (Theorem 3.24), and the finest such that algebraic K-theory is representable. Another reason is the *Brown-Gersten condition* (Definition 2.33 and Proposition 2.34), which provides a simpler characterization of Nisnevich-local simplicial presheaves (the latter will be closely related to fibrant objects in the theory we are developing). This is also called the Nisnevich descent theorem, and it plays an important role in the theory (see for example Proposition 2.38), so it is really a desirable feature of the Nisnevich topology. A similar statement is true for the Zariski topology. Furthermore, if S is Noetherian, the cohomological dimension (biggest dimension of a non-vanishing cohomology group for a sheaf on some fixed site) of (Sm_S, τ_{Nis}) is bounded by the Krull dimension of S, in contrast to the case of étale topology. This implies for example the compact generation of the corresponding stable category (also true under weaker assumptions, see [Hoy14]).

Morel and Voevodsky refer to the Nisnevich topology as an intermediate between the Zariski and étale topologies, having "the good properties of both while avoiding the bad ones". They also list some interesting properties in [MV99] (p 94-95). The Nisnevich topology seems to appear naturally in this context; for instance Morel claimed during his talk in München in 2018 (mentioned in the introduction), that he came up with the Nisnevich topology independently, when trying to solve his problem about representability of K-theory, without being previously aware of its existence. The Nisnevich topology was originally introduced by Yevsey Nisnevich in his (unpublished) thesis in 1982, in the context of adeles (he gives some details about this topology in [Nis89], in a context closer to our setting).

2.2.4 Hypercovers

Having now defined a suitable topology on Sm_S, we want to transfer some of its essence to the better-behaved category of simplicial presheaves over Sm_S. We will do so using the notion of hypercovers; the latter are generalizations of the concept of a Čech complex in topology. Following Dugger and Isaksen in [DI04], we first have a quick look at the topological side to develop some intuition.

Definition 2.14 (Čech complex)**.** Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. The *Čech complex* of this cover is the simplicial space whose n-th level is the topological space $\coprod_{i_0,\ldots,i_n\in I} U_{i_0}\cap\cdots\cap U_{i_n}$, with faces $\partial_j : U_{i_0}\cap\cdots\cap U_{i_n}\to U_{i_0}\cap\cdots\cap\widehat{U_{i_j}}\cap\cdots\cap U_{i_n}$ induced by the inclusions, and degeneracies $d_j : U_{i_0}\cap\cdots\cap U_{i_n}\to U_{i_0}\cap\cdots\cap U_{i_j}\cap U_{i_j}\cap\cdots\cap U_{i_n}$ induced by the identity.

Here, the 0-simplices represent the original cover. The 1-simplices are $\coprod_{i_0,i_1\in I} U_{i_0}\cap U_{i_1}$; note that $\{U_i\cap U_j\}_{i\in I}$ forms an open cover of U_j for all $j\in J$. Then, for all $j,k\in I$, $\{U_i\cap U_j\cap U_k\}_{i\in I}$ forms an open cover of $U_j\cap U_k$, which was an element in the cover we just described. And so on. A hypercover, in the case of topological spaces, intuitively consists in a refinement of this process. We will make this intuition more precise in what follows. We say that a continuous map is an *open covering map of* X if it is of the form $\coprod_{i\in I} U_i\to X$, induced by the inclusions, where $\{U_i\}_{i\in I}$ is an open cover of X. A hypercover will be a simplicial space $U_\bullet$ with a map to X viewed as a constant simplicial space, such that:

- the map $U_0\to X$ is a cover of X.
- heuristically, each U_n is a cover of the n-th level of a modified Čech complex built from the covers represented by the lower levels of $U_\bullet$.

To make this precise, let us "categorify" the notion of a Čech complex.

Definition 2.15 (Čech complex)**.** Let $(\mathcal{C},\tau)$ be a site (admitting finite products and pullbacks) and $\mathcal{U}:=\{f_i : U_i\to X\}_{i\in I}$ be a covering with respect to τ. Let $U=\coprod_{i\in I}\mathcal{C}(-,U_i)\in\mathsf{Pre}(\mathcal{C})$. Then the *Čech complex* $\check{C}(\mathcal{U})$ associated with $\mathcal{U}$ is the simplicial presheaf on $\mathcal{C}$ with n-th level given by the $(n+1)$-fold fiber product $\check{C}(\mathcal{U})_n = U\times_X\cdots\times_X U$ (where X is viewed as a presheaf) for all $n\in\mathbb{N}$.

For instance:

$$\check{C}(\mathcal{U})_1 = U\times_X U = \left(\coprod_{i\in I}\mathcal{C}(-,U_i)\right)\times_{\mathcal{C}(-,X)}\left(\coprod_{i\in I}\mathcal{C}(-,U_i)\right) = \coprod_{i,j\in I}\mathcal{C}(-,U_i\times_X U_j)$$

There is also an inductive description of the Čech complex coming from simplicial constructions. It is this description that we will generalize to define the "modified Čech complex" mentioned above, built from the first $(n-1)$ levels. Following [DI04] (modulo a small difference in terminology for the skeleton functor), we define:

Definition 2.16 (Skeleton and coskeleton, matching object)**.** Let $\mathcal{C}$ be a category with all finite limits and $n\in\mathbb{N}$. Consider the forgetful functor $U : {}_s\mathcal{C}\to {}_s\mathcal{C}_{\leq n}$, from simplicial objects in $\mathcal{C}$ to simplicial objects truncated at level n. Then it has a right adjoint $\mathsf{cosk}_n : {}_s\mathcal{C}_{\leq n}\to {}_s\mathcal{C}$ called the n*-th coskeleton functor*, and a left adjoint $\mathsf{sk}_n : {}_s\mathcal{C}_{\leq n}\to {}_s\mathcal{C}$ called the n*-th skeleton functor*. By abuse of notation, we also call cosk_n the composition $\mathsf{cosk}_n\circ U$, and similarly for sk_n. With this convention, there is an adjoint pair $\mathsf{sk}_n\dashv\mathsf{cosk}_n$ from ${}_s\mathcal{C}$ to itself.

Let $X\in\mathcal{C}$. The n*-th matching object* of $U_\bullet\in {}_s\mathcal{C}$ is defined as $M_n(U_\bullet):=(\mathsf{cosk}_{n-1}U_\bullet)_n$. The n*-th matching object over* $X\in\mathcal{C}$ of $U_\bullet\in({}_s\mathcal{C})\downarrow X$, where X is viewed as a constant simplicial object, is denoted by $M_n^X(U_\bullet)$ and is given by the same construction in the category $\mathcal{C}\downarrow X$ instead.

The skeleton functor produces a "free" object (freely fills all simplicial levels above n with degenerate simplices). The simplicial levels strictly higher than n consist only of the degeneracies of the lower-dimensional simplices. On the other hand, the n-th coskeleton functor is a "cofree" construction. It preserves the first n levels of the truncated simplicial set it is applied to, and adds an m-simplex for $m>n$ as soon as there is a compatible family of m-faces. More precisely, the n-th coskeleton of a truncated simplicial *set* $X_{\leq n}$ is characterized by the property that any map $\partial\Delta^k\to\mathsf{cosk}_n(X_{\leq n})$ extends uniquely to Δ^k for all $k>n$. We will see a justification of this fact in Remark 2.17 below. This property means that the $(k-1)$-th homotopy group is trivial (if X is a Kan complex). This fact will be crucial in the construction of Postnikov towers that we provide in Section 5.

To illustrate the definition of matching objects, we compute simplices up to level $n+1$ of the n-coskeleton of very simple examples. Only the non-degenerate simplices are represented. In colors blue and red are the maps from the boundary of a standard simplex that give rise to higher dimensional simplices in the coskeleton.

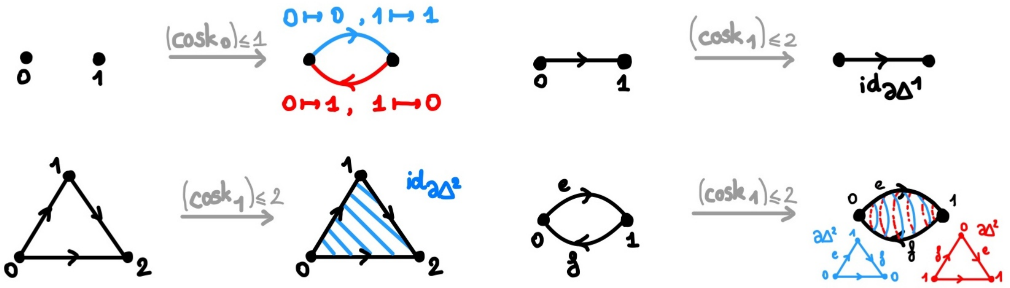


Figure 4: Truncations of the coskeleton of some small simplicial sets.

The term "matching object" comes from the terminology of Reedy categories (here, Δ^{op} with $\mathsf{deg}([n]) = n$) and diagrams indexed by a Reedy category (here, $\mathsf{Fun}(\Delta^{op}, \mathcal{C}) = {}_{\mathsf{S}}\mathcal{C}$). In this context, the matching object $M_n(U_\bullet)$ is defined as a limit of $U_\bullet$ over the matching category $\partial([n] \downarrow \Delta^{op})$. This coincides with Definition 2.16.

Remark 2.17. For our purposes, knowing a description of the matching objects for the category of presheaves on a site $\mathcal{C}$ will be sufficient. First note that ${}_{\mathsf{S}}\mathsf{Pre}(\mathcal{C}) \cong \mathsf{Pre}(\Delta \times \mathcal{C})$. In the setting of Definition 2.16, we have, for any object $X \in \mathcal{C}$ and $n \in \mathbb{N}^*$:

$$\begin{aligned} M_n(U_\bullet)(X) &= (\mathsf{cosk}_{n-1}(U_\bullet))_n(X) \\ &= \mathsf{cosk}_{n-1}(U_\bullet)([n], X) \\ &\cong {}_{\mathsf{S}}\mathsf{Pre}(\mathcal{C})([(\Delta \times \mathcal{C})(-, ([n], X))], \mathsf{cosk}_{n-1}(U_\bullet)) \\ &\cong {}_{\mathsf{S}}\mathsf{Pre}(\mathcal{C})_{\leq n-1}([(\Delta \times \mathcal{C})(-, ([n], X))]_{\leq n-1}, U_{\leq n-1}) \\ &\cong {}_{\mathsf{S}}\mathsf{Pre}(\mathcal{C})_{\leq n-1}(\partial\Delta^n \times \mathcal{C}(-, X), U_{\leq n-1}) \\ &\cong {}_{\mathsf{S}}\mathsf{Set}_{\leq n-1}(\partial\Delta^n, U_{\leq n-1}(X)) \end{aligned}$$

where the three first isomorphisms follow respectively from the Yoneda lemma, by adjunction, and by the fact that a representable object is viewed as a constant simplicial set. Similarly, by the Yoneda lemma, $U_n(X) \cong {}_{\mathsf{S}}\mathsf{Set}(\Delta^n, U_\bullet(X))$. Therefore, there is a natural map $U_n \to M_n(U_\bullet)$ given by restriction to the boundary of Δ^n (its $(n-1)$-th skeleton). In particular, this map is an isomorphism if and only if every map $\underline{\partial\Delta^n} \to U_\bullet$ has a unique extension to Δ^n. Running a similar argument is the case of simplicial sets gives the characterization of coskeleta discussed above. This also shows that coskeleton of simplicial presheaves can be characterized in a similar way.

This construction is relevant because we have the following result:

Lemma 2.18. *Let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be a covering in a site $(\mathcal{C}, \tau)$. Then $\check{C}(\mathcal{U})_n \cong M_n^X(\check{C}(\mathcal{U}))$.*

Proof. It suffices to show that $\check{C}(\mathcal{U})$ is an n-coskeleton for all $n \in \mathbb{N}$ (namely, the n-coskeleton of *some* simplicial object). Since the functor cosk_n preserves the first n simplicial levels, this will show that the Čech complex is necessarily its own n-th coskeleton for all $n \in \mathbb{N}$. By Remark 2.17, it suffices to show that every map $\underline{\partial\Delta^n} \to \check{C}(\mathcal{U})_\bullet$ has a unique extension to $\underline{\Delta^n}$. We show that, for all $Y \in \mathcal{C}$, every map $f : \partial\Delta^n \to \check{C}(\mathcal{U})_\bullet(Y)$ has a unique extension $\tilde{f}$ to Δ^n. If $\tilde{f}$ exists, it sends the non-degenerate n-simplex of Δ^n to some n-simplex in $\check{C}(\mathcal{U})_n(Y)$, namely to a morphism κ in $\mathcal{C}(Y, U_{i_0} \times_X \cdots \times_X U_{i_n})$, for some indices $i_0, \ldots, i_n \in I$. Consider the k-th face of Δ^n in $\partial\Delta^n$, namely the monotone injective map $[n-1] \to [n]$ omitting k, we denote it by g_k. In a similar manner, the $(n-1)$-simplices $f(g_k)$ for $0 \leq k \leq n$ are elements of $\mathcal{C}(Y, U_{i_{k,0}} \times_X \cdots \times_X U_{i_{k,n-1}})$. Since $\partial_0 g_0 = \partial_0 g_1$ in $\partial\Delta^n$, the same holds for $f(g_0)$ and $f(g_1)$, so we deduce $i_{0,k} = i_{1,k}$ for all $1 \leq k \leq n-1$, by construction of the faces map in the Čech complex. The following square is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{f(g_1)} & \begin{array}{c} U_{i_{1,0}} \times_X \cdots \times_X U_{i_{1,n-1}} \\ = \\ U_{i_0} \times_X U_{i_2} \times_X \cdots \times_X U_{i_n} \end{array} \\ {\scriptstyle f(g_0)}\downarrow & & \downarrow \\ \begin{array}{c} U_{i_{0,0}} \times_X \cdots \times_X U_{i_{0,n-1}} \\ = \\ U_{i_1} \times_X U_{i_2} \times_X \cdots \times_X U_{i_n} \end{array} & \longrightarrow & U_{i_2} \times_X \cdots \times_X U_{i_n} \end{array}$$

But since the pullback of the lower-right half of the square (in blue) is exactly $U_{i_0} \times_X \cdots \times_X U_{i_n}$, we get an induced map from Y to the latter object, which must be $\tilde{f}(\mathrm{id}_{[n]})$ by uniqueness in the universal property of the pushout. So if $\tilde{f}$ exists it is given in this way. Conversely, this construction yields an extension for f because it agrees with f on all the other faces, by applying the same argument that we did for the couple (g_0, g_1) to all pairs (g_i, g_j) in turns. □

To give a formal definition of hypercovers, we first need the analog of open covering maps for an arbitrary site:

Definition 2.19 (τ-covering maps)**.** Let $(\mathcal{C}, \tau)$ be a site.

- A map $f : X \to \mathcal{C}(-, c)$ in $\mathsf{Pre}(\mathcal{C})$ with target a representable object is called a τ-*covering map* if $X = \coprod_{i \in I} \mathcal{C}(-, x_i)$ is a coproduct of representable objects and the induced maps $\{f_i : x_i \to c\}_{i \in I}$ form a covering with respect to τ.
- A map $f : X \to Y$ in $\mathsf{Pre}(\mathcal{C})$ (with Y arbitrary this time) is called a τ-*covering map* if for any representable object $\mathcal{C}(-, c)$ with a map $g : \mathcal{C}(-, c) \to Y$, the base change of f to $\mathcal{C}(-, c)$ along g is a τ-covering map as defined in the previous bullet point.

Remark 2.20. Note that the two definitions agree, because by definition of a Grothendieck topology, coverings with respect to τ are stable under base change. So if we have a τ-covering map $f : \coprod_{i \in I} \mathcal{C}(-, x_i) \to \mathcal{C}(-, c)$ as in the first bullet point, and a map $\mathcal{C}(-, d) \to \mathcal{C}(-, c)$, then the base change is $\coprod_{i \in I} \mathcal{C}(-, x_i \times_c d) \to \mathcal{C}(-, d)$ with $\{x_i \times_c d \to d\}_{i \in I}$ a covering with respect to τ, so f also satisfies the requirements of the second bullet point.

We are finally ready to define hypercovers:

Definition 2.21 (Hypercover)**.** Let $(\mathcal{C}, \tau)$ be a site. Let $V \in \mathcal{C}$, viewed as an object of $_s\mathsf{Pre}(\mathcal{C})$. A map $f : U_\bullet \to V$ in $_s\mathsf{Pre}(\mathcal{C})$ is called a *hypercover* (with respect to τ) if:

- For all $n \in \mathbb{N}$, $U_n \in \mathsf{Pre}(\mathcal{C})$ is a coproduct of representable objects.
- The induced map $f_0 : U_0 \to V$ is a τ-*covering map*.
- For all $n \in \mathbb{N}^*$, the induced map $U_n \to M_n^V(U_\bullet)$ is a τ-covering map.

Remark 2.22. In [AM69] (Definition 8.4), an hypercovering over some site $(\mathcal{C}, \tau)$ with a final object is defined as an object $X_\bullet \in {}_s\mathcal{C}$, such that $X_0 \to *$ and $X_{n+1} \to M_n(X_\bullet)$ are covering maps for all $n \in \mathbb{N}^*$. The correspondence with Definition 2.21 above is obtained by applying the definition of [AM69] to the cite $\mathsf{Pre}(\mathcal{C}) \downarrow V$, where V is viewed as a representable object and the topology has covering the families $\{F_i \to G\}_{i \in I}$ such that the induced map $\coprod_{i \in I} F_i \to G$ is a τ-covering map.

Example 2.23. The Čech complex of a covering with respect to τ is a hypercover because the two first points in the definition of a hypercover hold by construction, and the third point follows directly from Lemma 2.18.

Example 2.24. For instance, in the big topological site (over the point), for a covering $\{U \to X, V \to X\}$, the beginning of the Čech complex looks like:

$$\ldots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathsf{Top}(-, U \cap U) \amalg \mathsf{Top}(-, U \cap V) \amalg \mathsf{Top}(-, V \cap U) \amalg \mathsf{Top}(-, V \cap V) \;\substack{\longrightarrow\\\longrightarrow}\; \mathsf{Top}(-, U) \amalg \mathsf{Top}(-, V)$$

Now, if $\{W\}_{j \in J}$ is an open cover of U, consider the cover $\mathcal{W}$ of X given by $\{W_j\}_{j \in J} \cup \{V\}$. If $\check{C}(\mathcal{W})_\bullet$ denotes the associated Čech complex, the following is a simple example of a hypercover of X:

$$\ldots \qquad \check{C}(\mathcal{W})_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \check{C}(\mathcal{W})_1 \;\substack{\longrightarrow\\\longrightarrow}\; \mathsf{Top}(-, U) \sqcup \mathsf{Top}(-, V)$$

Indeed, the 0th level has a τ-covering map to X by construction. All simplicial levels are coproducts of representable objects since all come from Čech complexes. Finally, the map from the first level of the complex to the first matching object is a τ-covering map by the explicit description given in Example 2.25 just below. Using the fact that we are dealing with open covers in Top, the second matching object is actually just $\check{C}(\mathcal{W})_2$ itself, because computing fiber products over U or V is the same as computing those fiber products over X: it's just the intersection in both cases. For higher degrees, note that the n-th matching objects for $n \geq 3$ of the complex above are just the matching objects for $\check{C}(\mathcal{W})_\bullet$ and hence the map from the n-th level to the n-th matching object is an isomorphism, in particular it is a τ-covering map. The fact that the matching objects can be computed in $\check{C}(\mathcal{W})_\bullet$ for higher level comes from the fact that the n-th matching object only depends on the two higher levels of the truncated simplicial set, i.e. levels $n-1$ and $n-2$. Indeed, the n-th level of the $(n-1)$-th coskeleton has one simplex for every map of $\partial\Delta^n$ into the truncated simplicial set; but as a simplicial set $\partial\Delta^n$ is the colimit of a diagram of copies of Δ^{n-1} and Δ^{n-2}. And maps from these standard simplices to a simplicial set just correspond to the $(n-1)$-simplices, respectively the $(n-2)$-simplices of the target. Whence our claim. Thus, the map from the n-th level to the n-th matching object is an isomorphism.

More generally, by the same kind of argument, replacing the levels of the Čech complex of some covering of X above a fixed index n by the same levels of a Čech complex for a covering of X that refines the original covering (replace a number of maps in the covering with coverings for the corresponding objects) always yields a hypercover of X. We can even repeat this process in a (possibly infinite) number of simplicial levels. This illustrates the slogan that hypercovers are refinements of Čech complexes.

Example 2.25. Let us make explicit the requirement in Definition 2.21 of $U_1 \to M_1^V(U_\bullet)$ being a τ-covering map. By definition, U_0 and U_1 can be written as $\coprod_{i\in I_0} \mathcal{C}(-, c_{0,i})$ and $\coprod_{i\in I_1} \mathcal{C}(-, c_{1,i})$ respectively. Since $U_0 = \check{C}(\{c_{0,i} \to V\}_{i\in I_0})_0$, we have

$$M_1^V(U_\bullet) = M_1^V(\check{C}(\{c_{0,i} \to V\}_{i\in I_0})) = \check{C}(\{c_{0,i} \to V\}_{i\in I_0})_1 = U_0 \times_V U_0 = \coprod_{i,j\in I_0} \mathcal{C}(-, c_{0,i} \times_V c_{0,j}).$$

Then for all $w \in \mathcal{C}$ and map $f : \mathcal{C}(-, w) \to U_0 \times_V U_0$, the map

$$\bar{f} : \coprod_{i\in I_1} \mathcal{C}(-, w) \times_{U_0 \times_V U_0} \mathcal{C}(-, c_{1,i}) \longrightarrow \mathcal{C}(-, w)$$

should be a τ-covering map. By the Yoneda lemma, f corresponds to the choice of $j, k \in I_0$ and an element of $\mathcal{C}(w, c_{0,j} \times_V c_{0,k})$. On the other hand, the natural map $U_1 \to M_1^V(U_\bullet)$ corresponds to the choice for all $i \in I_1$ of $j_i, k_i \in I_0$ and a morphism in $\mathcal{C}(c_{1,i}, c_{0,j_i} \times_V c_{0,k_i})$, by the same reasoning.

Then $\bar{f}$ is the map

$$\coprod_{i\in I_1 : (j_i, k_i) = (j,k)} \mathcal{C}(-, c_{1,i} \times_V w) \longrightarrow \mathcal{C}(-, w),$$

i.e. we require $\{c_{1,i} \times_V w \to w\}_{\{i\in I_1 | (j_i,k_i)=(j,k)\}}$ to be a covering with respect to τ. Because of the base change axiom, it amounts to checking this condition for $w \in \{c_{0,j} \times_V c_{0,k} \mid j, k \in I_0\}$. So the first simplicial level provides a covering for all fiber products of pairs of objects appearing in the 0th simplicial level.

In the case of topological spaces (respectively, a general site), this is very reminiscent of the short explanation right below Definition 2.14. Indeed, the topology of the disjoint union (respectively, the fact that we have a coproduct) forces the open covering map (the covering map) of the n-th matching object by the n-th level of a hypercover to consist in "individual" open covers (coverings) of each of the open sets (objects) appearing in the disjoint union (the coproduct) that forms the n-th matching object.

Following [AE17], we want to give a special role in the model category $_S\mathrm{Pre}(\mathrm{Sm}_S)$ to "sheaves up to homotopy" on Sm_S. They will be those objects that satisfy a hyperdescent condition (see Definition 2.31 below). To do so, we will choose a new model structure on Sm_S whose fibrant objects will be exactly the simplicial presheaves satisfying this hyperdescent condition (Lemma 2.32). It turns out that one can implement this idea by left Bousfield localization with respect to the hypercovers, in particular the latter will be added to the weak equivalences. This process creates a model structure such that weak equivalences of simplicial presheaves are not global anymore (i.e. defined objectwise), but rather local: they are "stalkwise weak equivalences". The latter are related to the model structure on simplicial presheaves constructed by Jardine; we will discuss this a bit more in subsection 6.1.

Definition 2.26 (Left Bousfield localization)**.** Given a simplicial model category $\mathcal{M}$ and $\mathcal{I}$ a set of maps in $\mathcal{M}$, let $\mathsf{map}(-,-)$ denote the simplicial function complexes in $\mathcal{M}$. Then:

- an object $X \in \mathcal{M}$ is called *$\mathcal{I}$-local* if it is fibrant and for all map $i \in \mathcal{I}$, the induced morphism $\mathsf{map}(i, X)$ is a weak equivalence in $_{S}\mathsf{Set}$.
- a morphism f in $\mathcal{M}$ is an *$\mathcal{I}$-local weak equivalence* if for any $\mathcal{I}$-local object Z, the induced morphism $\mathsf{map}(f, Z)$ is a weak equivalence in $_{S}\mathsf{Set}$.
- a *left Bousfield localization* $L_{\mathcal{I}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{I}$, if it exists, is the model structure on the underlying category of $\mathcal{M}$ whose weak equivalences are the $\mathcal{I}$-local weak equivalences, the cofibrations are the cofibrations of $\mathcal{M}$, and the fibrations are defined by the right lifting property.

Note that in particular, all morphisms in $\mathcal{I}$ become weak equivalences. The slogan for these definitions is that $\mathcal{I}$-local objects see every map in $\mathcal{I}$ as if it was a weak equivalence, and conversely, to "complete $\mathcal{I}$ into a class of equivalences", the $\mathcal{I}$-local weak equivalences are the maps that are seen as weak equivalences by all $\mathcal{I}$-local objects.

In general, a criterion for the existence of this localization is the following:

Proposition 2.27 (Existence of the Bousfield localization)**.** *The left Bousfield localization exists for any left proper, combinatorial, simplicial model category $\mathcal{M}$, with respect to any* set *$\mathcal{I}$ of morphisms in* M*.*

This localization is itself a left-proper, combinatorial, simplicial model category, whose fibrant objects are exactly the $\mathcal{I}$-local objects. Moreover, $\mathcal{I}$-local equivalences between $\mathcal{I}$-local objects are exactly the weak equivalences before localization between these two objects.

Proof. We give references for this proof. Lurie proves this statement when $\mathcal{I}$ consists of cofibrations in [Lur09], Proposition A.3.7.3. Barwick proves all claims except for the simplicial property in [Bar10], Theorem 4.7. Hirschhorn proves in [Hir03] (Theorem 4.1.1) that the Bousfield localization exists for left proper cellular categories, and that if the category is a simplicial model category, the same simplicial structure descends to its Bousfield localization. This last claim, once the existence of the Bousfield localization has been obtained, also holds in the case of left proper simplicial model categories, as can be checked by inspection of the proof.

The last part of our statement is a direct application of K. Brown's lemma to the right Quillen functor $\mathsf{id}_M : L_{\mathcal{I}}\mathcal{M} \to \mathcal{M}$ (see Remark 2.28 just below). □

Remark 2.28 (Bousfield localization yields a Quillen pair). If such a localization exists, then the identity functor on the underlying category M define a Quillen pair:

$$\mathsf{id}_M : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L_{\mathcal{I}}\mathcal{M} : \mathsf{id}_M$$

Indeed, it is clear that the identity functor $\mathsf{id}_M : \mathcal{M} \to L_{\mathcal{I}}\mathcal{M}$ preserves cofibrations, and it even preserves weak equivalences by Proposition 3.1.5 in [Hir03].

In particular, there is a *left* derived functor $\mathbb{L}\mathsf{id}_M : \mathcal{M} \to \mathsf{Ho}\,(L_{\mathcal{I}}(\mathcal{M}))$. As such, it can be computed on an object $X \in \mathcal{M}$ by applying the functor id_M to a cofibrant replacement of X with respect to the model structure before Bousfield localization.

But there is also a right derived functor computed by taking fibrant replacements in the localized model structure. Since an object of $L_{\mathcal{I}}(\mathcal{M})$ is by definition weakly equivalent to any of its fibrant replacements in $L_{\mathcal{I}}(\mathcal{M})$, in the presence of a fibrant replacement functor we may as well consider for our left derived functor, the post-composition of $\mathbb{L}\mathsf{id}_M$ by this fibrant replacement. It factors through $L_{\mathcal{I}}\mathcal{M}$.

Definition 2.29 (Localization functor). We define $L_{\mathcal{I}} : \mathcal{M} \to L_{\mathcal{I}}\mathcal{M}$ as the composition described just above.

As a left-derived functor for a left Quillen functor, it preserves homotopy colimits, for example. Another consequence of our definition is that for $X \in M$, the object $L_{\mathcal{I}}X$ is $\mathcal{I}$-local, and $\mathcal{I}$-locally weakly equivalent to X. Moreover, if X is already $\mathcal{I}$-local, then $L_{\mathcal{I}}(X)$ is weakly equivalent to X in the model structure *before* Bousfield localization. Indeed, X is weakly equivalent in this sense to its cofibrant replacement in $\mathcal{M}$, which is also a cofibrant replacement in $L_{\mathcal{I}}\mathcal{M}$, thus it is still I-local and taking the cofibrant replacement in $L_{\mathcal{I}}\mathcal{M}$ yields an $\mathcal{I}$-local weak equivalence of $\mathcal{I}$-local objects, and hence a weak equivalence in the original model structure.

2.2.5 Nisnevich hyperdescent

We now have the necessary tools to perform the advertised localization:

Proposition 2.30 (Localizing with respect to Nisnevich hypercovers)**.** *The left Bousfield localization* $L_{\mathsf{Nis}}({}_{S}\mathsf{Pre}(\mathsf{Sm}_S))$ *with respect to the class* Nis *of Nisnevich hypercovers exists.*

Proof. (sketch) We will use Proposition 2.27. The only difficulty is that a priori, the class Nis of hypercovers is not a set. To remedy this issue, we first prove that Sm_S is essentially small. Let $X \in \mathsf{Sm}_S$ be a smooth scheme of finite type over S with structure morphism $f : X \to S$. Then by Definition 1.4 and the finite-type assumption, there exist affine covers $X = \bigcup_{i\in I} U_i$ and $S = \bigcup_{i\in I} V_i$ with $f^{-1}(V_i) = U_i$ and $U_i \to V_i$ a standard smooth map for all $i \in I$. There is only a set of open affine covers of S (a subset of 2^{2^S}). Note that if $V_i = \mathsf{Spec}(R)$ is fixed, there is only a set of possibilities up to isomorphism for standard smooth maps (determined by the choice of non-negative integers n and r, and the choice of r polynomials in $R[x_1, \ldots, x_{n+r}]$). So X must be a gluing of the U_i's, determined by the choice of open sets $U_{ij} \subseteq U_i$ for all $i, j \in I$ and gluing maps $U_{ij} \to U_{ji}$ for all $i, j \in I$. Since collections of subsets of fixed sets are sets, and collections of functions between two fixed sets are again sets, all steps above conserve the smallness of the collections. So Sm_S is essentially small. Therefore, we may consider only those hypercovers with target belonging to a *set* a representatives for isomorphism classes of smooth schemes over S.

Still, it remains to check that the collection of hypercovers $U_\bullet \to V$ with target a fixed representable presheaf V is essentially small. We will check that up to isomorphism there is only a set of possibilities for each simplicial level of $U_\bullet$. Then, it follows that there is only a set of possibilities for faces, degeneracies, and the structure map to V. By definition of a hypercover, for all $n \in \mathbb{N}^*$, U_n must be isomorphic to a coproduct of representable objects such that the natural map $U_n \to M_n^V(X)$ is a Nisnevich covering map, and $U_0 \to V$ is a Nisnevich covering map. Once the target is fixed, there is only a set of such covering maps up to isomorphism, since Sm_S is essentially small. Then, inductively, there is only a set of possibilities for $U_\bullet$ and its maps to V (still up to isomorphism).

Then, we have to check that the category ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)$ with the projective model structure is left-proper, simplicial and combinatorial. We already mentioned the fact that this model structure is simplicial, referring to [BK72] (p 314). Combinatoriality follows from Proposition A.2.8.2 in [Lur09] and the fact that the Quillen model structure on ${}_{S}\mathsf{Set}$ is itself combinatorial (stated in [Lur09], section A.2.7). For left-properness, note that ${}_{S}\mathsf{Set}$ is left-proper (as a model category with all objects cofibrant, using Proposition A.2.4.2. in [Lur09]), and therefore so is ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)$ (Remark A.2.8.4 in [Lur09]). □

Note that this is more general and can be applied to any Grothendieck topology on Sm_S or on an essentially small site, see Theorem 3.33 in [AE17].

A very similar construction can also be performed in ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)_*$, adding disjoint basepoints where needed. Covering maps are just defined as pointed maps that are sent to covering maps of unpointed presheaves by the forgetful functor (and coproducts are now levelwise wedge sums instead of levelwise disjoint unions).

Step 2

The model category $\mathsf{Spc} := L_{\mathsf{Nis}}({}_{S}\mathsf{Pre}(\mathsf{Sm}_S))$ with weak equivalences the Nisnevich-local weak equivalences, projective cofibrations, and fibrations defined by the right lifting property.
(Respectively, its pointed analog Spc_*)

As advertised earlier, the fibrant objects of this new model structure are very interesting: they are "homotopy sheaves" with respect to the Nisnevich topology. Let us make more precise what this means.

Definition 2.31 (Nisnevich descent, space)**.** We say that a simplicial presheaf X on Sm_S satisfies *Nisnevich hyperdescent* if for every Nisnevich hypercover $U \to V$, with $U_n \cong \coprod_{i \in I_n} \mathsf{Sm}_S(-, R_{n,i})$ for all $n \in \mathbb{N}$ and $V \in \mathsf{Sm}_S$, the natural map $X(V) \to \mathsf{holim}_\Delta \left([n] \mapsto \prod_{i \in I_n} X(R_{n,i})\right)$ is a weak equivalence of simplicial sets. We abbreviate this homotopy limit as $\mathsf{holim}_n\, X(U)$.

A fibrant object in $\mathsf{Spc} = \mathsf{L}_{\mathsf{Nis}}\left({}_S\mathsf{Pre}(\mathsf{Sm}_S)\right)$ is called a *space*.

Lemma 2.32 (Nisnevich-local objects are homotopy sheaves)**.** *A simplicial presheaf X on Sm_S is a space if and only if it takes its values in Kan complexes and satisfies Nisnevich hyperdescent.*

Proof. By Proposition 2.27, the fibrant objects in the Bousfield localization are exactly the Nisnevich local objects, i.e. fibrant objects of ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ such that any hypercover $U \to V$ induces a weak equivalence of simplicial sets $\mathsf{map}(V, X) \to \mathsf{map}(U, X)$. Since fibrations are defined objectwise in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$, a simplicial presheaf is fibrant in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ if and only if it is objectwise fibrant, i.e. if and only if it takes its values in Kan complexes.

Let $U \to V$ be a Nisnevich hypercover of $V \in \mathsf{Sm}_S$ (viewed as the simplicial presheaf it represents). By the Yoneda lemma, $\mathsf{map}(V, X) \cong X(V)$ (more details in Remark 2.44). We now want to identify $\mathsf{map}(U, X)$ as the homotopy limit in 2.31.

We claim that U is the homotopy colimit over Δ^{op} of the simplicial object in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ (seen as a functor $\Delta^{\mathsf{op}} \to {}_S\mathsf{Pre}(\mathsf{Sm}_S)$) associating to $[n] \in \Delta^{\mathsf{op}}$ the constant simplicial presheaf U_n, for all $n \in \mathbb{N}$. This simplicial object can be seen as a bisimplicial presheaf, whose diagonal is U. So our claim is a generalization to simplicial presheaves of a classical statement for simplicial sets (see the page "Bisimplicial set" in the nLab, Proposition 2.4).

Let us first see how the claim implies the result of the lemma. We wanted to identify the homotopy limit in the statement with $\mathsf{map}(U, X)$. But, by our claim:

$$\mathsf{map}(U, X) \simeq \mathsf{map}(\mathsf{hocolim}_{[n] \in \Delta^{\mathsf{op}}} U_n, X) \simeq \mathsf{holim}_{[n] \in \Delta} \mathsf{map}(U_n, X)$$

because $\mathsf{map}(-, X)$ is right Quillen on ${}_S\mathsf{Pre}(\mathsf{Sm}_S)^{\mathsf{op}}$ by the axioms of a simplicial model category.

Using the levelwise description of U, for all $n \in \mathbb{N}$, we have:

$$\mathsf{map}(U_n, X) = \mathsf{map}\left(\coprod_{i \in I_n} \mathsf{Sm}_S(-, R_{n,i}), X\right) = \prod_{i \in I_n} \mathsf{map}(\mathsf{Sm}_S(-, R_{n,i}), X) = \prod_{i \in I_n} X(R_{n,i})$$

(see Remark 2.44 for the last equality), so the lemma follows.

We now turn to the proof of our claim. By a general formula expressing homotopy colimits as coends (Theorem 7.1 on the page "Homotopy limit" in the nLab), we have:

$$\mathsf{hocolim}_{[n] \in \Delta^{\mathsf{op}}} U_n = \int^{[n] \in \Delta^{\mathsf{op}}} Q_{\mathsf{inj}}(*) \times Q_{\mathsf{proj}}(U_n)$$

where Q_{inj} is a cofibrant replacement in the category of cosimplicial objects in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$, with the injective model structure, and Q_{proj} is a cofibrant replacement in the category of simplicial objects in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$, with the projective model structure. The hypotheses we have to verify for the formula to hold are the fact that ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ is a combinatorial simplicial model category, which was stated in Subsection 2.2.5, and that $U_\bullet : \Delta^{\mathsf{op}} \to {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ is a simplicially enriched functor when Δ^{op} has the trivial enrichment. The second fact holds because the hom-objects in Δ^{op} are constant simplicial sets, and all objects in the image of $U_\bullet$ are constant as simplicial objects too.

Consider $\underline{\Delta^\bullet}$, the cosimplicial object in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ associating to $[k] \in \Delta$ the simplicial constant presheaf $\underline{\Delta^k}$. It is cofibrant in the injective model structure, because $\underline{\Delta^k}$ is cofibrant in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ (see the end of Remark 2.36). Moreover, $U_\bullet$ is cofibrant in the projective model structure on simplicial objects in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$. Indeed, the latter category also has the projective model structure, so we may just consider the projective model structure on the category of functors $\Delta^{\mathsf{op}} \times \mathsf{Sm}_S^{\mathsf{op}} \to {}_S\mathsf{Set}$, i.e. simplicial presheaves on $\Delta \times \mathsf{Sm}_S$. As we will see later in Remark 2.36, the cofibrant objects are the levelwise coproducts of representables that split into a degenerate and non-degenerate part.

Therefore, since U_n is a constant simplicial object, given levelwise by a coproduct of representable presheaves, it is cofibrant. Whence:

$$\int^{[n]\in\Delta^{op}} Q_{\mathsf{inj}}(*) \times Q_{\mathsf{proj}}(U_n) = \int^{[n]\in\Delta^{op}} \underline{\Delta^n} \times U_n$$

Finally, using the Yoneda lemma (viewing simplicial presheaves on Sm_S as presheaves on $\Delta \times \mathsf{Sm}_S$), to conclude that the coend above is isomorphic to U, we compute, for any simplicial presheaf Y:

$$\begin{aligned} {}_S\mathsf{Pre}(\mathsf{Sm}_S)\left(\int^{[n]\in\Delta^{op}} \underline{\Delta^n} \times U_n, Y\right) &\cong \int^{[n]\in\Delta^{op}} {}_S\mathsf{Pre}(\mathsf{Sm}_S)\,(\underline{\Delta^n} \times U_n, Y) \\ &\cong \int^{[n]\in\Delta^{op}} \mathsf{Pre}(\mathsf{Sm}_S)\,(U_n, Y_n) \\ &\cong {}_S\mathsf{Pre}(\mathsf{Sm}_S)(U, Y) \end{aligned}$$

where we have used some properties of coends, that can be found as Corollary 3.2 (for commuting coend and hom-sets) and Proposition 4.1 (for the last isomorphism, viewing our objects as functors on Δ^{op}) on the nlab page "End". We have followed the proof of Example 5.1. The second isomorphism follows from similar computations as in Remark 4.2, and amounts to using the mapping space adjunction and Yoneda's lemma. □

Descent is actually a wider concept present in algebraic geometry. In general, a *descent problem* is about *descending* a property from one base scheme to another (i.e. transfer it along a morphism instead of pulling it back, as for base changes), for example from the algebraic closure of $\overline{k}$ of a field k to k itself.

There are also several *descent theorems*. For a choice of a topology τ on some category of schemes, consider a covering family $\{p_i : X_i \to X\}$ with respect to τ. If X is a scheme, let $\mathsf{QCoh}(X)$ be the collection (monoid) of quasi-coherent sheaves of modules on X. We are interested in understanding the image of the map:

$$\mathsf{QCoh}(X) \xrightarrow{\prod_i (p_i)_*} \prod_i \mathsf{QCoh}(X_i)$$

induced by pullback, namely answering the question of when a collection of quasi-coherent sheaves $\mathcal{F}_i \in \mathsf{QCoh}(X_i)$ can be glued into a quasi-coherent sheaf on X. In many well-behaved topologies, such as Zariski, étale, fppf, fpqc (for this case, which is the finest topology of the four, see [Gro95]), the answer is that the collection belongs to the image of the map introduced above if it comes with *descent data*, i.e. isomorphisms between the pullback of $\mathcal{F}_i$ to $X_i \times_X X_j$ and the pullback of $\mathcal{F}_j$ to $X_j \times_X X_i$ for all i and j, that satisfy a certain cocycle condition on "triple intersections" (further pullback to another X_k), making the following into an equalizer:

$$\mathsf{QCoh}(X) \xrightarrow{\prod_i (p_i)_*} \prod_i \mathsf{QCoh}(X_i) \underset{\prod (p_k)_*}{\overset{\prod (p_j)_*}{\rightrightarrows}} \prod_{j,k} \mathsf{QCoh}(X_j \times_X X_k).$$

This is actually like asking the presheaf of monoids $\mathsf{QCoh}(-)$ to be a sheaf with respect to τ. In this equalizer we recognize something that looks like the beginning of $\mathsf{QCoh}(-)$ applied to the Čech complex of $\{X_i \to X\}_i$. This gives a vague intuition about why imposing Nisnevich hyperdescent is some kind of sheafification requirement on our simplicial presheaves, but up to "homotopy only" (because of the weak equivalence and the homotopy limit in the Nisnevich descent condition).

In our situation, representable objects will be Nisnevich fibrant (Proposition 2.35), so in particular they will satisfy Nisnevich hyperdescent. This is an analog for simplicial presheaves of the fact that Nisnevich topology is subcanonical (Proposition 2.13).

As an illustration of a similar hyperdescent condition in a different context, Dugger and Isaksen prove in [DI04] that the map $\mathsf{hocolim}\, U_\bullet \to X$ is a weak equivalence of topological spaces, for any topological hypercover $U_\bullet \in {}_S\mathsf{Top}$ of a topological space X. The arrows are reversed compared to Nisnevich hyperdescent condition because this happens directly in the category Top (instead of embedding topological spaces into the category of presheaves).

When motivating the choice of the Nisnevich topology, we already mentioned the existence of a "simple" criterion characterizing spaces, i.e. fibrant objects in $L_{\mathsf{Nis}}({}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S))$. This criterion relies on the Brown-Gersten condition:

Definition 2.33 (Heuristic definition of the Brown-Gersten condition)**.** A simplicial presheaf on $(\mathcal{C}, \tau)$ satisfies the *Brown-Gersten condition*, also called *excision*, if it satisfies a Mayer-Vietoris type property: fiber product squares of well-chosen pairs of morphisms in $\mathcal{C}$ (in particular, forming coverings with respect to τ) should be carried by this presheaf to homotopy pullback squares.

A topology τ has the Brown-Gersten property if for objectwise fibrant simplicial presheaves, the homotopy (hyper)descent property (i.e. the $\mathcal{I}$-locality condition with respect to the collection $\mathcal{I}$ of τ-hypercovers) is equivalent to the Brown-Gersten condition stated above. Namely, it suffices to check descent for certain simple covers made of two morphisms. This property holds in particular for some topologies over schemes, topological spaces or smooth manifolds. We now specialize Definition 2.33 to our situation:

Theorem 2.34 (Criterion for fibrancy: the Nisnevich descent theorem)**.** *If* S *is Noetherian of finite Krull dimension, then* $Y \in \mathsf{Spc}$ *a (non-empty) presheaf of Kan complexes is fibrant (is a space) if and only if for every elementary Nisnevich square:*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & \lrcorner & \big\downarrow p \\ U & \xrightarrow[\iota]{} & X \end{array}$$

the map obtained after applying the (contravariant) functor Y*:*

$$Y(X) \longrightarrow Y(U) \times^h_{Y(U \times_X V)} Y(V)$$

is a weak equivalence of simplicial sets, and $Y(\emptyset) \simeq *$*. Here* $\times^h$ *denotes a homotopy pullback.*

Proof. The proof is rather long and involved, so we defer it to Theorem A.8 in the Appendix. $\square$

This condition corresponds to checking Nisnevich hyperdescent for Nisnevich coverings coming from elementary Nisnevich squares only. Indeed, if we consider the hypercover $\check{C}(\mathcal{W})_\bullet$ of X with $\mathcal{W}$ the Nisnevich covering $\{U \to X, V \to X\}$, then the condition of Y satisfying hyperdescent for this specific hypercover (Definition 2.31) amounts to the condition in Proposition 2.34 (see in the Appendix, in the proof of Theorem A.8).

Proposition 2.35 ((Co)fibrancy of representable objects)**.** *Given any scheme* X *over* S*, not necessarily smooth, consider* X *as a constant simplicial presheaf* $\mathsf{Hom}_S(-, X)$ *on* Sm_S*.*

- *(i) If* $X \in \mathsf{Sm}_S$*, then* X *viewed as an object of* ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ *is Nisnevich local (namely it is fibrant in* Spc*), and cofibrant.*
- *(ii) If* S *is Noetherian of finite Krull dimension, then* X *is Nisnevich local (even for non-smooth* X*).*

Proof. We delay the proof of the fibrancy part of (i) to subsection A.2 in Appendix 7, right after Theorem A.5. Representable objects are cofibrant in ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$, Spc or $\mathsf{Spc}_{\mathbb{A}^1}$ (which we will define soon) indifferently: indeed, all three categories have the same cofibrations, and we can show cofibrancy in ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$ by checking by hand the left lifting property. Let $f : W \xrightarrow{\sim} \!\!\!\!\rightarrow Y$ be an acyclic fibration of simplicial presheaves, and consider a map $g : X \to Y$. By Yoneda's lemma, a map $\ell : X \to W$ lifting g corresponds to a 0-simplex $\tilde{\ell} \in W_0(X)$ such that $f(X)(\tilde{\ell}) = \tilde{g}$ the zero-simplex in $Y_0(X)$ that represents the map g. Since $f(X) : W(X) \to Y(X)$ is an acyclic fibration of simplicial sets by definition, is has the right lifting property with respect to the cofibration of simplicial sets $\emptyset \to *$, so there is a lift of the map $* \to Y(X)$ given by $\tilde{g}$ along $f(X)$ to a map $* \to W(X)$. Then a suitable choice for $\tilde{\ell}$ is given by the image of the zero simplex of $*$ by this lift. Hence $\mathsf{Sm}_S(-, X)$ is a cofibrant simplicial presheaf.

For the proof of (ii), we first note that, as a constant simplicial object, $\mathsf{Hom}_S(-, X)$ is in particular valued in Kan complexes. If we wanted to check fibrancy using Lemma 2.32, we would have to check that for any Nisnevich hypercover $U_\bullet \to V$, where V is representable by a smooth scheme over S, the natural map

$$\mathsf{Hom}_S(V, X) \to \mathsf{holim}_{[n]\in\Delta} \prod_{i\in I_n} \mathsf{Hom}_S(R_{n,i}, X) = \mathsf{lim}_{[n]\in\Delta} \mathsf{Hom}_S\left(\coprod_{i\in I_n} R_{n,i}, X\right)$$

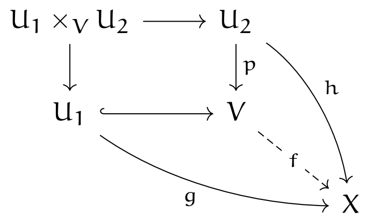
is a weak equivalence of simplicial sets. In other words, we would have to check that $\mathsf{Hom}_S(-, X)$ is a sheaf with respect to Nisnevich *hypercovers*. If $U_\bullet$ was a Čech complex, this would just be the definition of a sheaf, but up to homotopy. However, the homotopy limit is a strict limit here since representable presheaves take their values in discrete simplicial sets. For more about this, see Remark 2.37 just below this example. Here, Proposition 2.34 makes it easier to check this fact: we just have to show that if $\{U_1 \to V, U_2 \to V\}$ is a Nisnevich covering forming a Nisnevich square, then

$$X(V) \to X(U_1) \times^h_{X(U_1\times_V U_2)} X(U_2)$$

is a weak equivalence of simplicial sets. Since the simplicial sets on the right are constant ones, the homotopy limit is just a usual limit of sets (it is possible to show that such a diagram is injectively fibrant, by showing the lifting property by hand). Thus we want

$$\mathsf{Hom}_S(V, X) \cong \mathsf{Hom}_S(U_1, X) \times_{\mathsf{Hom}_S(U_1\times_V U_2, X)} \mathsf{Hom}_S(U_2, X)$$

as sets via the natural map; which amounts to checking that a Nisnevich square is also a pushout of schemes. Given a diagram as follows:



we aim at constructing a unique morphism f of schemes over S. This argument is the one from Lemma 81.9.1 in the Stacks project, Tag 0DVH. The induced map $U_1 \amalg U_2 \to V$ is easily verified to be an étale cover (a surjective étale map). The key property here is that $\mathsf{Hom}_S(-, X)$ is a sheaf for the étale topology (the étale topology on the site of *all* schemes over S is subcanonical, see for example the Stacks project, Tag 03NV, Remark 59.15.9). Therefore, we have an equalizer:

$$\mathsf{Hom}(V, X) \longrightarrow \mathsf{Hom}(U_1 \amalg U_2, X) \underset{-\pi_2\to}{\overset{-\pi_1\to}{}} \mathsf{Hom}((U_1 \amalg U_2) \times_V (U_1 \amalg U_2), X)$$

with the first map injective (whence the uniqueness). We have an induced map $g \amalg h : U_1 \amalg U_2 \to X$, so it suffice to check that $(g \amalg h) \circ \pi_1 = (g \amalg h) \circ \pi_2$ if and only if the square formed by g, h and the projections above is commutative. We have

$$(U_1 \amalg U_2) \times_V (U_1 \amalg U_2) = (U_1 \times_V U_1) \amalg (U_1 \times_V U_2) \amalg (U_2 \times_V U_1) \amalg (U_2 \times_V U_2).$$

Since U_1 is Zariski open in V, $U_1 \times_V U_1 = U_1$ maps identically to U_1. Moreover, f and g agree on $U_1 \times_V U_2 \cong U_2 \times_V U_1$ if and only if the square above is commutative. Finally, for $U_2 \times_V U_2$, we have a surjective étale morphism

$$\ell : U_2 \amalg ((U_1 \times_V U_2) \times_{U_1} (U_1 \times_V U_2)) \to U_2 \times_V U_2$$

(étale maps are stable under base change). Indeed, writing $V = U_1 \cup V \setminus U_1$, over $V \setminus U_1$ the fiber product is just $p^{-1}(V \setminus U_1)$, since $p|_{p^{-1}(V\setminus U_1)}$ is an isomorphism onto $V \setminus U_1$ by hypothesis. Thus U_2 surjects onto this part. Over U_1 the fiber product is exactly the second component of the disjoint union. Now, since g and h agree on $U_1 \times_V U_2$, the precompositions of $(g \amalg h) \circ \pi_1$ and $(g \amalg h) \circ \pi_2$ by ℓ agree, and thus these two maps agree on $U_2 \times_V U_2$. This concludes the proof. □

Remark 2.36 (Projectively cofibrant presheaves). More generally, cofibrant objects are exactly those simplicial presheaves X such that X is levelwise a coproduct of representable objects, with X_n being a coproduct of two presheaves of sets, the first one consisting of all degenerate simplices (see [Dug01b] and the MathOverflow post [hgb]). These references mention retracts of representable objects, but in our case, retracts of representable objects are themselves representable: if $\iota : F \to \mathsf{Sm}_S(-, X)$ admits a retraction $r : \mathsf{Sm}_S(-, X) \to F$ with F a simplicial presheaf and X a scheme, then F is representable by the pullback of $X \xrightarrow{\Delta} X \times_S X \xleftarrow{\iota \circ r} X$).

Moreover any cofibration of simplicial sets $A \hookrightarrow B$, namely a levelwise injective map, induces a projective cofibration $\underline{A} \to \underline{B}$ between the corresponding simplicial constant presheaves. Indeed, consider an objectwise acyclic fibration $X \to Y$ between two presheaves, and maps $\underline{A} \to X$, $\underline{B} \to Y$ as to form a commutative square. Since $X(S) \to Y(S)$ is an acyclic fibration of simplicial sets, we may find a lift $B \to X(S)$ in the square formed by A, B, $X(S)$ and $Y(S)$. Then, for any scheme $U \in \mathsf{Sm}_S$, we choose as a map $B \to X(U)$ the composition of $B \to X(S)$ and of the image by X of the structure map $U \to S$ of U. This yields a lift in the diagram of presheaves because S (with the identity as structure map) is a terminal object in Sm_S.

Remark 2.37 (Hypercovers versus covers). Instead of asking for hyperdescent (descent for hypercovers), one can ask for simple *descent*, namely we consider only those hypercovers that are Čech complexes. It turns out that the two conditions coincide for *hypercomplete* sites, and that Sm_S is hypercomplete when S is Noetherian of finite Krull dimension. But there are other conditions under which sheaves are hypersheaves, namely satisfy hyperdescent. One of them is stated as Theorem A.5; in particular sheaves of sets, seen as simplicial sheaves, are hypersheaves (in this situation, the homotopy limit becomes a strict limit).

The criterion of Proposition 2.34 can be seen as a requirement of preserving homotopy pushouts (or pullbacks, because of contravariance) in view of the following proposition:

Proposition 2.38 (Elementary Nisnevich squares are homotopy pushouts)**.** *If S is Noetherian of finite Krull dimension, then any elementary Nisnevich square is a homotopy pushout when viewed as a diagram of simplicial presheaves in* $L_{\mathsf{Nis}}({}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S))$.

Proof. Given a Nisnevich square for $\{\iota : U \to X, p : V \to X\}$ as in Definition 2.11, pick a factorization of the pullback map $U \times_X V \to U$ as a cofibration to some object Y, followed by an acyclic fibration in Spc (where Y might not be representable anymore). Then the pushout Q in Spc of $V \longleftarrow U \times_X V \hookrightarrow Y$ computes the homotopy pushout of $V \longleftarrow U \times_X V \longrightarrow U$ in Spc, by proposition 3.2. Namely, we have a diagram as follows:

$$\begin{array}{ccccc} U \times_X V & \hookrightarrow & Y & \xrightarrow{\sim} & U \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\iota} \\ V & \longrightarrow & Q & \overset{f}{\dashrightarrow} & X \end{array} \qquad (p : V \to X)$$

where the dashed map f is obtained from the universal property of the pushout. To show that X is the homotopy pushout of $V \longleftarrow U \times_X V \longrightarrow U$ in Spc it therefore suffices to show that f is a Nisnevich local weak equivalence. By definition we have to prove that for any $Z \in$ Spc Nisnevich local, the morphism of function complexes $\mathsf{map}(f, Z) : \mathsf{map}(X, Z) \to \mathsf{map}(Q, Z)$ is a weak equivalence of simplicial sets. Applying the functor $\mathsf{map}(-, Z)$ to the diagram above, we obtain:

$$\begin{array}{ccccc} \mathsf{map}(U \times_X V, Z) & \longleftarrow & \mathsf{map}(Y, Z) & \xleftarrow{\sim} & \mathsf{map}(U, Z) \\ \uparrow & & \uparrow & & \uparrow \\ \mathsf{map}(V, Z) & \longleftarrow & \mathsf{map}(Q, Z) & \xleftarrow{\mathsf{map}(f,Z)} & \mathsf{map}(X, Z) \end{array}$$

Since Z is Nisnevich local, $\mathsf{map}(U, Z) \to \mathsf{map}(Y, Z)$ is a weak equivalence of simplicial sets. By right properness of ${}_{\mathsf{S}}\mathsf{Set}$, it suffices to show that the right hand side square is a homotopy pullback. This follows from the reverse pasting law for (homotopy) pullbacks: the outer rectangle is a homotopy pullback by Proposition 2.34, since we had an elementary Nisnevich square and Z is fibrant. The left-hand side square is a homotopy pullback as $\mathsf{map}(-, Z)$ a right Quillen functor $(\mathsf{Spc})^{\mathsf{op}} \to {}_{\mathsf{S}}\mathsf{Set}$, therefore it preserves homotopy limits (corresponding to homotopy colimits in Spc). Indeed, since Spc is a simplicial model category, this functor has a left adjoint (sending a simplicial set K to the presheaf $U \in \mathsf{Sm}_S \mapsto \mathsf{map}_{{}_{\mathsf{S}}\mathsf{Set}}(K, Z(U))$), and $\mathsf{map}(-, Z)$ sends (acyclic) fibrations in $(\mathsf{Spc})^{\mathsf{op}}$ (namely (acyclic) cofibrations in Spc) to (acyclic) fibrations in ${}_{\mathsf{S}}\mathsf{Set}$: this is exactly axiom M7 in the definition of a simplicial model category (see [Hir03], Definition 9.16), applied to the (acyclic) cofibration in question in Spc and the fibration $Z \twoheadrightarrow *$. This concludes the proof. □

This provides for example a nice "excision property": given $U \in \mathsf{Sm}_S$ a smooth scheme, $Y \subseteq U$ an open subscheme, containing a closed subscheme Z, we have a Nisnevich elementary square:

$$\begin{array}{ccc} Y \setminus Z & \longrightarrow & U \setminus Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & U \end{array}$$

which is a homotopy pushout in $\mathsf{Spc}_{\mathbb{A}^1}$ by the previous proposition. Then, by general properties of homotopy pushouts, the homotopy cofibers of the two vertical maps are weakly equivalent (in $\mathsf{Spc}_{\mathbb{A}^1}$), namely:

$$Y/(Y \setminus Z) \simeq U/(U \setminus Z).$$

2.3 Contractibility of the affine line

The model category Spc still does not encode a notion of $\mathbb{A}^1$-invariance:

Example 2.39. Using the material of Section 4, a quick computation shows that $\pi_0^{\mathbb{N}}(\mathbb{A}^1) \cong \mathbb{A}^1$ is not trivial as a sheaf of sets (but higher homotopy is trivial), and thus $\mathbb{A}^1$ is not weakly equivalent to the point in Spc (we use the zero section $S \to \mathbb{A}^1$ as a basepoint for $\mathbb{A}^1 \in \mathsf{Spc}_*$). Indeed, for $k \geq 0$, $\pi_k^{\mathsf{Nis}}(\mathbb{A}^1)$ is the sheafification of the presheaf mapping $U \in \mathsf{Sm}_S$ to $[\mathcal{S}^n \wedge U_+, \mathbb{A}^1]_{\mathsf{Spc}_*}$. As a representable object, $\mathbb{A}^1$ is Nisnevich fibrant (Proposition 2.35), and we will see in Section 4 that $\mathcal{S}^n \wedge U_+$ is cofibrant. The derived smash product when seen as a functor of ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ also gives the derived smash product of Spc_* because the functor $\mathrm{id} : {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* \to \mathsf{Spc}_*$ is left Quillen, so it preserves homotopy pushouts (and wedge sums). Therefore, we can compute (see Remark 4.2 for the details):

$$[\mathcal{S}^n \wedge U_+, A^1]_{{}_S\mathbb{P}(\mathbb{S}_S),*} \cong [\mathcal{S}^n, \mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S),*}(U_+, \mathbb{A}^1)]_{{}_S\mathsf{Pre}(\mathsf{Sm}_S),*} \cong [\mathcal{S}^n, \mathsf{Sm}_S(U, \mathbb{A}^1)]_{{}_S\mathsf{Set},*}$$

because $\mathbb{A}^1$ is already fibrant so the mapping space is computed levelwise. With $\mathsf{Sm}_S(U, \mathbb{A}^1)$ being a constant simplicial sheaf, the homotopy groups at a fixed based points are trivial in positive dimension, and the presheaf of Nisnevich connected components is therefore given by $\mathsf{Sm}_S(-, \mathbb{A}^1)$, which is already a sheaf since Nisnevich topology is subcanonical (stated as Proposition 2.13).

Another argument is that $\mathbb{A}^1 \to *$ is not a Nisnevich-local equivalence: indeed, since both $\mathbb{A}^1$ and $*$ are Nisnevich-local, this would implies that the map is an objectwise weak equivalence, but $\mathsf{Sm}_S(S, \mathbb{A}^1) \neq *$, a contradiction.

To impose $\mathbb{A}^1$-homotopy invariance, we proceed to a second Bousfield localization: we want to turn the map $\mathbb{A}^1 \to *$ into a weak equivalence, but also we will ask for the projection $X \times_S \mathbb{A}^1 \to X$ to be a weak equivalence for all smooth schemes X. Since we want a set (and not a proper class) $\mathcal{I}$ of maps to ensure the existence of the Bousfield localization, we choose $\mathcal{I}$ to be the collection of all projections $X \times_S \mathbb{A}^1 \to X$ where X ranges over representatives for isomorphism classes in Sm_S. This is indeed a set as we saw in the proof of Proposition 2.30. This localization exists by Proposition 2.27, because the same proposition ensure that $L_{\mathsf{Nis}}({}_S\mathsf{Pre}(\mathsf{Sm}_S))$ is still a left-proper, combinatorial, simplicial model category. The $\mathcal{I}$-local weak equivalences will be called $\mathbb{A}^1$*-local equivalences*.

For the pointed version of the construction, we choose the basepoint of $\mathbb{A}^1$ as a representable to be the map $* = \mathsf{Sm}_S(-, S) \to \mathsf{Sm}_S(-, \mathbb{A}^1)$ induced by the zero section $S \to \mathbb{A}^1$. We then perform a Bousfield localization with respect to the projections $U_+ \times \mathbb{A}^1 \to \mathbb{A}^1$ for U varying over a set of representatives for isomorphism classes in Sm_S.

Thus we obtain:

Step 3

The model category $\mathsf{Spc}_{\mathbb{A}^1} := L_{\mathbb{A}^1} L_{\mathsf{Nis}}{}_S\mathsf{Pre}(\mathsf{Sm}_S)$ with weak equivalences the $\mathbb{A}^1$-local weak equivalences, projective cofibrations, and fibrations defined by the right lifting property.

(Respectively, its pointed analog $\mathsf{Spc}_{\mathbb{A}^1,*}$)

Definition 2.40 ($\mathbb{A}^1$-homotopy category)**.** The $\mathbb{A}^1$*-homotopy category* is the homotopy category of the model category $\mathsf{Spc}_{\mathbb{A}^1}$.

Remark 2.41 (All projections are $\mathbb{A}^1$-weak equivalences). For any simplicial presheaf X, the projection $X \times \mathbb{A}^1 \to X \times S = X$ is an $\mathbb{A}^1$-local weak equivalence. A priori we only inverted such projections for X a smooth scheme. This makes it easier for example to check $\mathbb{A}^1$-locality of a given object. But since every simplicial presheaf $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ is a homotopy colimit of representable objects ([Dug01b], Proposition 2.8 and below), the property extends to all of $\mathsf{Spc}_{\mathbb{A}^1}$. To see the first fact, by taking a cofibrant replacement of X in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$, we may assume that X is cofibrant. Then X is levelwise a coproduct of representable presheaves (Remark 2.36), but also $X \simeq \mathsf{hocolim}_{[n]\in\Delta^{op}} X_n$, where X_n is the constant simplicial presheaf given by the n-simplices of X (proof of Lemma 2.32), namely a coproduct of representable objects. So given $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$, let us write $X \simeq \mathsf{hocolim}_i\, U_i$ for some $U_i \in \mathsf{Sm}_S$, for all i in a small category $\mathcal{I}$. For any $\mathbb{A}^1$-local object Z, we have:

$$\begin{aligned}
\mathsf{map}(X, Z) &\simeq \mathsf{map}(\mathsf{hocolim}_i\, U_i, Z) && \text{(equivalences in } {}_S\mathsf{Pre}(\mathsf{Sm}_S) \text{ are local weak equivalences)}\\
&\simeq \mathsf{holim}_i\, \mathsf{map}(U_i, Z) \\
&\simeq \mathsf{holim}_i\, \mathsf{map}(U_i \times_S \mathbb{A}^1, Z) && \text{(since Z is } \mathbb{A}^1\text{-local and } U_i \in \mathsf{Sm}_S)\\
&\simeq \mathsf{map}(\mathsf{hocolim}_i\, (U_i \times_S \mathbb{A}^1), Z) \\
&\simeq \mathsf{map}(X \times \mathbb{A}^1, Z)
\end{aligned}$$

where the last line comes from the fact that the functor $- \times Y$ commutes with homotopy colimits in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ for any cofibrant object Y. Indeed, we check that this functor preserves colimits, cofibrations and acyclic cofibrations. Since colimits are computed levelwise and objectwise, the preservation of colimits is just a statement about colimits of sets, which we know is true. The preservation of a cofibration, respectively acyclic cofibration $A \hookrightarrow B$ follows from the pushout-product axiom (from the monoidal model structure) applied to the latter (acyclic) cofibration and $\emptyset \hookrightarrow Y$. Alternatively, one could observe that objectwise weak equivalences are preserved because homotopy groups of simplicial sets commute with finite products.

Fibrant objects in Spc were called spaces. Now, we define:

Definition 2.42 ($\mathbb{A}^1$-space)**.** A fibrant object of $\mathsf{Spc}_{\mathbb{A}^1}$ is called an $\mathbb{A}^1$*-space*.

Here is a rephrasing of the second condition in the definition of $\mathbb{A}^1$-local objects:

Definition 2.43 ($\mathbb{A}^1$-invariant, $\mathbb{A}^1$-rigid)**.** A simplicial presheaf $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ is called $\mathbb{A}^1$*-invariant* if $X(U) \to X(U \times_S \mathbb{A}^1)$ induced by the projection $U \times_S \mathbb{A}^1 \to U$ is a weak equivalence of simplicial sets for all $U \in \mathsf{Sm}_S$. An S-scheme X (not necessarily smooth) which is $\mathbb{A}^1$-invariant when considered as a representable presheaf, is called an $\mathbb{A}^1$*-rigid scheme*.

Indeed, with this definition:

Remark 2.44. Being $\mathbb{A}^1$-local is equivalent to being Nisnevich fibrant and $\mathbb{A}^1$-invariant. Indeed, it suffices to check that $X(U) \to X(U \times_S \mathbb{A}^1)$ (as in the definition above) is a weak equivalence of simplicial sets if and only if $\mathsf{map}(U, X) \to \mathsf{map}(U \times_S \mathbb{A}^1, X)$ is a weak equivalence of simplicial sets.

The k-simplices of $\mathsf{map}(U, X)$ are given by:

$$\begin{aligned}
{}_S\mathsf{Pre}(\mathsf{Sm}_S)(U \times \underline{\Delta}^k, X) &= \mathsf{Pre}(\mathsf{Sm}_S \times \Delta)([(\mathsf{Sm}_S \times \Delta)(-, (U, [0])) \times (\mathsf{Sm}_S \times \Delta)(-, (S, [k]))], X) \\
&\cong X(S \times_S U, [0] \times [k]) \\
&= X_k(U),
\end{aligned}$$

and therefore it coincides with $X(U)$ as a simplicial set. This argument also shows that the same holds true for $X(U \times_S \mathbb{A}^1)$, and under this identification the maps induced by the projection coincide. Whence our claim.

With the computation we just did, in particular we can describe fibrant objects of $\mathsf{Spc}_{\mathbb{A}^1}$ as follows: a simplicial presheaf $X \in \mathsf{Spc}_{\mathbb{A}^1}$ is fibrant if and only if it takes its values in Kan complexes, satisfies Nisnevich hyperdescent, and for all $U \in \mathsf{Sm}_S$, the map $X(U) \to X(\mathbb{A}^1 \times_S U)$ induced by the projection is a weak equivalence of simplicial sets.

Indeed, by Proposition 2.27, X is fibrant if and only if it is $\mathbb{A}^1$-local. The condition of being fibrant in Spc is equivalent to being valued in Kan complexes and satisfying Nisnevich hyperdescent by Proposition 2.32.

Further properties of $\mathbb{A}^1$-invariant simplicial presheaves and $\mathbb{A}^1$-rigid schemes can be found in subsection 3.3.

Example 2.45 (Not all representable objects are $\mathbb{A}^1$-spaces). Recall that representable presheaves are Nisnevich fibrant. So to be $\mathbb{A}^1$-fibrant, a representable presheaf only has to be $\mathbb{A}^1$-invariant (Remark 2.44). However, this does not hold for all representable objects. For instance, $\mathbb{A}^1$ itself is not $\mathbb{A}^1$-invariant as a presheaf: otherwise, the projection would induce a bijection of sets $j : \mathsf{Sm}_S(S, \mathbb{A}^1) \to \mathsf{Sm}_S(\mathbb{A}^1, \mathbb{A}^1)$ by definition, but then it means that every map of S-schemes $\mathbb{A}^1 \to \mathbb{A}^1$ factors through S (via the projection). This is clearly not the case for the identity of $\mathbb{A}^1$.

Here is a summary of the different constructions we encountered:

Model category	Weak equivalences	Fibrations	Cofibrations	Fibrant objects	Cofibrant objects
$_s\mathsf{Set}$	Maps whose geometric realization is a weak homotopy equivalence	Defined by the right lifting property	Levelwise monomorphisms	Kan complexes (horn-filling condition)	All
$_s\mathsf{Pre}(\mathsf{Sm}_S)$	Objectwise weak equivalences of simplicial sets	Objectwise fibrations of simplicial sets	Projective cofibrations: defined by the left lifting property	Simplicial presheaves of Kan complexes	X a levelwise coproduct of representables, such that X_n is a coproduct of two presheaves of sets, the first one consisting of all degenerate simplices
Spc	Nisnevich-local weak equivalences	Defined by the right lifting property	Projective cofibrations	Spaces: simplicial presheaves of Kan complexes satisfying Nisnevich hyperdescent	Same as for $_s\mathsf{Pre}(\mathsf{Sm}_S)$
$\mathsf{Spc}_{\mathbb{A}^1}$	$\mathbb{A}^1$-local weak equivalences	Defined by the right lifting property	Projective cofibrations	$\mathbb{A}^1$-spaces: fibrant objects of Spc that are $\mathbb{A}^1$-invariant	Same as for $_s\mathsf{Pre}(\mathsf{Sm}_S)$

3 Some constructions and examples

We now interest ourselves to some more or less concrete computations and examples. We will reproduce some classical topological constructions, such as spheres, suspensions, loop spaces, and Thom spaces in the motivic context, namely in $\mathsf{Spc}_{\mathbb{A}^1}$, or more precisely $\mathsf{Spc}_{\mathbb{A}^1,*}$. We will also say more about $\mathbb{A}^1$-invariance, $\mathbb{A}^1$-rigidity and encounter the singular space functor.

3.1 Smash products, loop spaces and suspensions

Inspired by topology, in any pointed model category $\mathcal{M}$, we can make the following constructions:

Definition 3.1 (Smash product, loop space and suspension functors)**.**

- The *smash product* $X \wedge Y$ of two objects $X, Y \in \mathcal{M}$ is the pushout of $* \longleftarrow X \vee Y \longrightarrow X \times Y$ (where $\vee$ denotes the coproduct in $\mathcal{M}$ since $\mathcal{M}$ is pointed)
- The *(reduced) suspension* ΣX of an object $X \in \mathcal{M}$ is given by the homotopy cofiber of $X \longrightarrow *$, or equivalently as the homotopy pushout of $* \longleftarrow X \longrightarrow *$.
- The *loop space* ΩX of an object $X \in \mathcal{M}$ is given by the homotopy fiber of $* \longrightarrow X$, or equivalently as the homotopy pullback of $* \longrightarrow X \longleftarrow *$.

In particular, all these constructions make sense in the pointed category $\mathsf{Spc}_{\mathbb{A}^1,*}$. Since colimits of simplicial presheaves are computed objectwise, the smash product of $X, Y \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ is given by $(X \wedge Y)(U) = X(U) \wedge Y(U)$ (smash product as pointed simplicial sets) for all $U \in \mathsf{Sm}_S$.

Let us recall how to compute homotopy pullbacks and pushouts:

Proposition 3.2 (Computation of homotopy pushouts/pullbacks, Proposition A.2.4.4 in [Lur09])**.** *In the model category $\mathcal{M}$, the homotopy pushout/pullback of $A \longleftarrow B \longrightarrow C$, respectively $A \longrightarrow B \longleftarrow C$, can be computed:*

(i) As the strict pushout/pullback of the same diagram with $A \longleftarrow B$, respectively $A \longrightarrow B$, replaced by a cofibration/fibration, and B and C replaced by cofibrant/fibrant objects.

(ii) if $\mathcal{M}$ is left/right proper, as the strict pushout of the same diagram with one of the legs replaced by a cofibration/fibration (no need to replace the objects as well).

Note that ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$, Spc and $\mathsf{Spc}_{\mathbb{A}^1}$ (and their pointed companions) are all left and right proper. Left properness is a consequence of Theorem 2.27 (and the proof of Proposition 2.30). Right properness of ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ follows in the same way, from that of ${}_S\mathsf{Set}$. For $\mathsf{Spc}_{\mathbb{A}^1}$, and therefore for Spc, it appears as Lemma 2.2 in [DRØ03]. For Spc, see [MV99], Section 2, Theorem 2.7 (and the comparison theorems between the model structures in subsection 6.1).

Since smash product and mapping spaces are adjoint at the level of simplicial sets, they induce a self adjunction $X \wedge - \dashv \mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)_*}(X, -)$ of ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$, where smash products and mapping spaces are computed objectwise. In particular, for all $X, Y \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ and $U \in \mathsf{Sm}_S$, we have $\mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)_*}(X, Y)(U) = \mathsf{map}_{{}_S\mathsf{Set},*}(X(U), Y(U))$, and for all $n \in \mathbb{N}$, the n-simplices are $\mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)_*}(X, Y)(U)_n = {}_S\mathsf{Set}_*(X(U) \wedge (\Delta^n)_+, Y(U))$. The index "${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$" in the notation above is to distinguish the mapping spaces from the simplicial function complexes $\mathsf{map}(X, Y) \in {}_S\mathsf{Set}$.

These functors define a Quillen pair with respect to the projective model structure since fibrations and weak equivalences are defined levelwise, and the adjunction is already a Quillen pair at the level of simplicial sets.

Proposition 3.3 (Monoidal model structure for $\mathsf{Spc}_{\mathbb{A}^1,*}$ (Section 2.1 and Lemma 2.20 in [DRØ03]))**.** *For all $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$, there are Quillen pairs:*

$$X \wedge - : {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* \rightleftarrows {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* : \mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)_*}(X, -)$$
$$X \wedge - : \mathsf{Spc}_* \rightleftarrows \mathsf{Spc}_* : \mathsf{map}_{\mathsf{Spc},*}(X, -)$$
$$X \wedge - : \mathsf{Spc}_{\mathbb{A}^1,*} \rightleftarrows \mathsf{Spc}_{\mathbb{A}^1,*} : \mathsf{map}_{\mathsf{Spc}_{\mathbb{A}^1},*}(X, -)$$

making ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$, Spc and $\mathsf{Spc}_{\mathbb{A}^1,*}$ into monoidal model categories.

Additionally, smash product preserves weak equivalences in $\mathsf{Spc}_{\mathbb{A}^1,*}$.

In particular, there are left and right derived functors

$$\mathbb{L}(X \wedge -) : {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* \to \mathsf{Ho}\left({}_S\mathsf{Pre}(\mathsf{Sm}_S)_*\right)$$
$$\mathbb{R}(\mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)_*}(X,-)) : {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* \to \mathsf{Ho}\left({}_S\mathsf{Pre}(\mathsf{Sm}_S)_*\right)$$

computed on the objects by applying the original functors to a cofibrant, respectively fibrant replacement. These functors factor through ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ and the localization, therefore we obtain functors $\mathbb{L}(X \wedge -) : {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* \to {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ and $\mathbb{R}(\mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)_*}(X,-)) : {}_S\mathsf{Pre}(\mathsf{Sm}_S)_* \to {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$. The story is similar in the cases of the categories Spc_* and $\mathsf{Spc}_{\mathbb{A}^1,*}$. Although the non-derived functors are the same for all three categories, a difference might appear when deriving them.

We will not expand on the matter, but the three categories ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$, Spc, and $\mathsf{Spc}_{\mathbb{A}^1}$ (and their pointed analogs) admit functorial fibrant and cofibrant replacements. It is a standard fact that functorial replacements exists for simplicial sets; in particular we get a functorial fibrant replacement functor for ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ because fibrations are defined objectwise. The case of cofibrant replacements is discussed in [Dug01b], section 2.6. Since cofibrant objects coincide in the three categories, this answers the question for all of them. More generally, combinatorial model categories have functorial fibrant and cofibrant replacements (see for example [Dug01a]); and all three model structures are combinatorial (Theorem 2.27).

Warning

From now on, every time we will write a mapping space (with values in a category of presheaves), we will refer to the *derived functor* with respect to the corresponding model structure. All smash products will be derived with respect to $\mathsf{Spc}_{\mathbb{A}^1,*}$.

If we really want to denote their non-derived version, we will write $X \wedge^o -$ and $\mathsf{map}^o \mathbb{A}^1, *(X,-)$ respectively (with "o" as in "objectwise"). In particular, with this convention:

Lemma 3.4 (Smash product as a homotopy pushout)**.** *Let* $X, Y \in \mathsf{Spc}_{\mathbb{A}^1,*}$. *Their (derived) smash product* $X \wedge Y$ *is the homotopy pushout of* $* \longleftarrow X \vee Y \longrightarrow X \times Y$.

Proof. Let $\mathcal{D}$ be the span category $\bullet \longleftarrow \bullet \longrightarrow \bullet$. The strict smash product $X \wedge^o -$ is the functor $\mathsf{Spc}_{\mathbb{A}^1,*} \to \mathsf{Fun}\left(\mathcal{D}, \mathsf{Spc}_{\mathbb{A}^1,*}\right) \to \mathsf{Spc}_{\mathbb{A}^1,*}$ obtained by composition of the functor which to $Y \in \mathsf{Spc}_{\mathbb{A}^1,*}$ associates the span $* \longleftarrow X \vee Y \longrightarrow X \times Y$ and the functor $\mathsf{colim}_{\mathcal{D}}$ ($\mathsf{Spc}_{\mathbb{A}^1,*}$ admits all colimits). Since the composition of derived functors provides a derived functor for the composite (we endow the category $\mathsf{Fun}\left(\mathcal{D}, \mathsf{Spc}_{\mathbb{A}^1,*}\right)$ with the projective model structure), we get that the derived smash product can be computed as the composition of the functor associating to a simplicial presheaf Y the span $* \longleftarrow X \vee QY \longrightarrow X \times QY$, where QY is a cofibrant replacement of Y in $\mathsf{Spc}_{\mathbb{A}^1,*}$ (left-derived functors are computed on objects by taking cofibrant replacement) with the functor $\mathsf{hocolim}_{\mathcal{D}}$. Since homotopy pushouts are invariant under weak equivalences, we get that $X \wedge Y$ is also a homotopy pushout for $* \longleftarrow X \vee Y \longrightarrow X \times Y$ (since QY is weakly equivalent to Y and wedge sums and products preserve weak equivalences in our setting). $\square$

In particular, this respects the usual symmetry of the smash product. More precisely: whether we consider the first or second variable as fixed (i.e., $X \wedge Y$ as $(X \wedge -)(Y)$ or $(- \wedge Y)(X)$) before taking the derived functor, the results obtained will be weakly equivalent because in both cases they are the homotopy pushout of the same diagram.

We also make the following definition:

Definition 3.5 (Quotient spaces in $\mathsf{Spc}_{\mathbb{A}^1}$)**.** Given $f : X \to Y$ a morphism in $\mathsf{Spc}_{\mathbb{A}^1}$, we define the *quotient space* Y/X as the homotopy cofiber of f.

Since quotients do not exist in general in the category of (smooth) schemes, whenever we write a quotient of schemes, we are viewing them as representable presheaves and apply Definition 3.5.

3.2 Motivic spheres

The circle S^1 appear everywhere in topology. Can we define what a motivic circle is? As it turns out, there are two answers, because motivic homotopy theory gathers simplicial sets and schemes.

Let us first mention the two intervals. In the introduction, we talked about $\mathbb{A}^1$ playing the role of the interval in $\mathbb{A}^1$-homotopy theory. But since $\mathsf{Spc}_{\mathbb{A}^1}$ is a category of *simplicial* valued presheaves, there is another candidate to the title of interval: the constant presheaf $\underline{\Delta^1}$ with value the simplicial interval Δ^1. Similarly, there are two contestants to the title of motivic circle: the simplicial circle $\mathcal{S}^1$, i.e. the constant simplicial presheaf with value the circle $S^1 := \Delta^1 \amalg_{\partial\Delta^1} \Delta^0 \in {}_S\mathsf{Set}$, or a circle that comes from algebraic geometry: the group scheme $\mathbb{G}_m$.

The latter choice also makes sense topologically, because we will see just below that its suspension is, up to weak equivalence, the projective line $\mathbb{P}^1$. If we work over $\mathbb{C}$, the complex points of the projective line give the Riemann sphere, which is itself a two-dimensional sphere S^2 topologically speaking. The topological 2-sphere can also be obtain by collapsing the circle in the plane; an analog in algebraic geometry to this would be to "quotient out" $\mathbb{A}^1_{\mathbb{C}}$ (the complex plane) by $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ (whose $\mathbb{C}$-points deformation retracts onto the circle). Precisely:

Lemma 3.6 ($\mathbb{G}_m$ is a circle)**.** *The following objects are $\mathbb{A}^1$-weakly equivalent (in $\mathsf{Spc}_{\mathbb{A}^1,*}$):*

$$\Sigma(\mathbb{G}_m, 1) \simeq (\mathbb{P}^1, \infty) \simeq \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$$

Proof. The basepoint 1 for $\mathbb{G}_m$ corresponds to the 1-section $S \to \mathbb{A}^1 \setminus \{0\}$. The quotient is naturally pointed by the image of $\mathbb{A}^1 \setminus \{0\}$. The point at infinity in $\mathbb{P}^1$ is the complement of the image of $\mathbb{A}^1$ via a chosen standard inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. We just have to prove that the following square is a homotopy pushout in $\mathsf{Spc}_{\mathbb{A}^1,*}$, where the two maps from $\mathbb{A}^1$ to $\mathbb{P}^1$ are the "standard charts":

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

where the maps $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ are the inclusions. Indeed, if this holds, since homotopy pushouts are by design invariant under weak equivalences, $\mathbb{P}^1$ is also the homotopy pushout of the cospans: $S \longleftarrow \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1$ and $S \longleftarrow \mathcal{A}^1 \setminus \{0\} \longrightarrow S$, since $\mathcal{A}^1 \cong \mathcal{A}^1 \times_S S \longrightarrow S$ is an $\mathbb{A}^1$-weak equivalence by construction. By definition, these are the homotopy cofiber of $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$, i.e. the quotient $\mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$; respectively the suspension of $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$.

This square is an elementary Nisnevich square (the two maps $\mathbb{A}^1 \to \mathbb{P}^1$ even form a Zariski cover). Thus it is a homotopy pushout in $\mathsf{L}_{\mathsf{Nis}}({}_S\mathsf{Pre}(\mathsf{Sm}_S))$ by Proposition 2.38. It is also a homotopy pushout in $\mathsf{Spc}_{\mathbb{A}^1,*}$, because the identity on the underlying categories $\mathsf{Spc}_* \to \mathsf{Spc}_{\mathbb{A}^1,*}$ is left-Quillen and left-Quillen functors preserve homotopy colimits. ☐

The object $\mathbb{G}_m \in \mathsf{Spc}_{\mathbb{A}^1}$ is called the *Tate circle*. It is not $\mathbb{A}^1$-equivalent to the simplicial circle $\mathcal{S}^1$ in general: indeed, as we will see in Example 3.11 below, if S is reduced then $\mathbb{G}_m$ is rigid and hence by the results in subsection 4.1, $\pi_n^{\mathbb{A}^1}(\mathbb{G}_m)$ is trivial for all $n \geq 1$ and $\pi_0^{\mathbb{A}^1}(\mathbb{G}_m)$ is weakly equivalent to $\mathbb{G}_m$ as a Nisnevich sheaf. On the other hand, $\pi_0^{\mathbb{A}^1}(\mathcal{S}^1)$ is the sheafification of $[U_+, \mathcal{S}^1]_{\mathbb{A}^1}$. If we take a fibrant replacement S' of S^1 in ${}_S\mathsf{Set}_*$, the corresponding simplicial constant presheaf $\underline{S'}$ is fibrant in $\mathsf{Spc}_{\mathbb{A}^1}$ and thus $[U_+, \mathcal{S}^1]_{\mathsf{A}^1} \cong [S^0, \underline{S'}(U)]_{{}_S\mathsf{Set},*} \cong \pi_0(S') \cong \pi_0(S^1)$ is trivial. In particular it is already a sheaf. The Nisnevich homotopy sheaves are the same; but $\mathbb{G}_m$ is not isomorphic to the trivial sheaf (not even $\mathbb{A}^1$-weakly equivalent: by Example 3.12, this would imply that $\mathbb{G}_m$ is isomorphic to S as a scheme). Since $\mathbb{A}^1$-equivalences induce isomorphisms (and a bijection in dimension 0) of $\mathbb{A}^1$-homotopy sheaves (Proposition 4.3), $\mathcal{S}^1$ is not $\mathbb{A}^1$-equivalent to $\mathbb{G}_m$.

In topology, the (reduced) suspension functor as defined in subsection 3.1 can be computed by smashing with the circle S^1. There is a similar result in the motivic setting:

Lemma 3.7 (Suspension as a smash product)**.** *The suspension of $X \in \mathsf{Spc}_{\mathbb{A}^1,*}$ as defined in subsection 3.1, i.e. the homotopy cofiber of $X \to *$, is $\mathbb{A}^1$-equivalent to the smash product $\mathcal{S}^1 \wedge X$.*

Proof. The smash product is a left Quillen functor, therefore it preserves homotopy pushouts. If we show that $\mathcal{S}^1$ is the homotopy pushout of $* \longleftarrow \mathcal{S}^0 \longrightarrow *$, then $X \wedge \mathcal{S}^1$ is the homotopy pushout of $* \wedge X \longleftarrow \mathcal{S}^0 \wedge X \longrightarrow * \wedge X$, equivalently of $* \longleftarrow X \longrightarrow *$ (because $*$ and $\mathcal{S}^0$ are already cofibrant in $\mathsf{Spc}_{\mathbb{A}^1,*}$ (Remark 2.36), so the derived smash products are strict).

Since $\underline{\Delta}^1$, the constant simplicial presheaf with value the standard 1-simplex, is weakly equivalent to the point in ${}_s\mathsf{Pre}(\mathsf{Sm}_S)$ (because it is already contractible as a simplicial set, and weak equivalences are defined levelwise), it remains so in $\mathsf{Spc}_{\mathbb{A}^1,*}$. Thus the homotopy pushout of $* \longleftarrow \mathcal{S}^0 \longrightarrow *$ is equivalently the homotopy pushout of $* \longleftarrow \mathcal{S}^0 \longrightarrow \underline{\Delta}^1$. This can be computed as a strict pushout by proposition 3.2, because $\mathcal{S}^0 \longrightarrow \underline{\Delta}^1$ is a cofibration: it is map of constant simplicial presheaves and the morphism is induced by a levelwise monomorphism in ${}_s\mathsf{Set}$ (Remark 2.36). Finally, the strict pushout of this diagram is by definition $\mathcal{S}^1$, so we are done.

□

But instead we may also want to take smash products with the Tate circle, and we obtain a different suspension functor. Inspired by the topological fact that $(\mathbb{S}^1)^{\wedge n} \simeq \mathbb{S}^n$, we will now use the two different motivic circles to obtain bigraded spheres and suspension functors:

Definition 3.8 (Bigraded suspension and spheres)**.** For all $a \geq b \geq 0$, we define the (a,b)-*suspension functor* $\Sigma^{a,b} : \mathsf{Spc}_{\mathbb{A}^1,*} \to \mathsf{Spc}_{\mathbb{A}^1,*}$ as the functor $\mathbb{G}_m^{\wedge b} \wedge (\mathcal{S}^1)^{\wedge(a-b)} \wedge -$. The (a,b)-*sphere* is then defined as $\mathcal{S}^{a,b} := \Sigma^{a,b}(\mathcal{S}^0) \cong \mathbb{G}_m^{\wedge b} \wedge (\mathcal{S}^1)^{\wedge(a-b)}$.

In this notation, the functor Σ of subsection 3.1 rewrites $\Sigma^{1,0}$. The choice of the grading comes from notation in motivic cohomology.

One might hope that such central objects as spheres would have simpler descriptions. One possible notion of a "nice description" for objects in $\mathsf{Spc}_{\mathbb{A}^1}$ is to have a *smooth model*, i.e. a weakly equivalent representable object coming from Sm_S. For now, the only spheres for which smooth models have been found are those of the form $\mathcal{S}^{2n-1,n}$ and $\mathcal{S}^{2n,n}$ (when S is affine). It is even conjectured that these are the only ones (see [ADF17], where it is also proven that some motivic spheres *do not* have smooth models, using representability of motivic cohomology).

Proposition 3.9 (Smooth models for motivic spheres)**.**

(i) For all $n \geq 1$, there is an $\mathbb{A}^1$-weak equivalence $\mathcal{S}^{2n-1,n} \simeq \mathbb{A}^n \setminus \{0\}$.

(ii) For all $n \geq 1$, there is an $\mathbb{A}^1$-weak equivalence $\mathcal{S}^{2n,n} \simeq \mathbb{A}^n / (\mathbb{A}^n \setminus \{0\})$.

(iii) If $S = \mathsf{Spec}(k)$ is the spectrum of a field, then for all $n \geq 1$, there is another smooth model for $\mathcal{S}^{2n-1,n}$: there is an $\mathbb{A}^1$-weak equivalence

$$\mathcal{S}^{2n-1,n} \simeq \mathsf{Spec}\left(k[x_1,\dots,x_n,y_1,\dots,y_n] \big/ \left(\textstyle\sum_{i\leq n} x_i y_i - 1\right)\right)$$

(iv) If $S = \mathsf{Spec}(A)$ is affine, then for all $n \geq 1$, there is an $\mathbb{A}^1$-weak equivalence

$$\mathcal{S}^{2n,n} \simeq \mathsf{Spec}\left(A[x_1,\dots,x_n,y_1,\dots,y_n,z] \big/ \left(\textstyle\sum_{i\leq n} x_i y_i - z(1+z)\right)\right)$$

Proof. This argument is based on the proof on Proposition 4.40 and Corollaries 4.41 and 4.46 in [AE17]. It doesn't even really contain more details, but we reproduce it here because it provides interesting examples of computations in $\mathsf{Spc}_{\mathbb{A}^1,*}$.

The proof of (i) is by induction. For $n = 1$, we have $\mathcal{S}^{1,1} = \mathbb{G}_m \cong \mathbb{A}^1 \setminus \{0\}$ by definition. Assume that the statement holds for some $n \in \mathbb{N}^*$ fixed; then

$$\mathcal{S}^{2n+1,n+1} \simeq \mathbb{G}_m \wedge \mathcal{S}^1 \wedge \mathcal{S}^{2n-1,n} \simeq \mathbb{G}_m \wedge \mathcal{S}^1 \wedge (\mathbb{A}^n \setminus \{0\})$$

since by Proposition 3.3, smash product preserves weak equivalences in $\mathsf{Spc}_{\mathbb{A}^1,*}$. By Lemma 3.7, this triple smash product can be computed as the homotopy cofiber of $\mathbb{G}_m \wedge (\mathbb{A}^n \setminus \{0\}) \to *$. We have four commutative squares:

$$\begin{array}{ccccccc}
* & = & * & = & * & & * \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{A}^n \setminus \{0\} & \xleftarrow{* \vee \mathrm{id}_{\mathbb{A}^n \setminus \{0\}}} & \mathbb{G}_m \vee (\mathbb{A}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}_{\mathbb{G}_m} \vee *} & \mathbb{G}_m & & * \\
\| & & \downarrow & & \| & & \downarrow \\
\mathbb{A}^n \setminus \{0\} & \xleftarrow{* \times \mathrm{id}_{\mathbb{A}^n \setminus \{0\}}} & \mathbb{G}_m \times (\mathbb{A}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}_{\mathbb{G}_m} \times *} & \mathbb{G}_m & & \mathbb{A}^{n+1} \setminus \{0\} \\
 & & & & & & \\
* & \longleftarrow & \mathbb{G}_m \wedge (\mathbb{A}^n \setminus \{0\}) & \longrightarrow & * & &
\end{array}$$

where the dark blue spans are the column-wise, respectively row-wise *homotopy* pushouts. For the homotopy pushout of the second column, we use Lemma 3.4. For the homotopy pushout of the second row, write $X = \mathbb{A}^n$ and $Y = \mathbb{G}_m$, then the homotopy pushout of $X \longleftarrow X \vee Y \longrightarrow Y$ is equivalent to $*$: indeed this is given by the following square:

$$\begin{array}{ccccccc}
* & \longleftarrow & Y & = & Y & & * \\
\| & & \uparrow & & \uparrow & & \| \\
* & = & * & = & * & & * \\
\downarrow & & \downarrow & & \| & & \| \\
X & = & X & \longrightarrow & * & & * \\
 & & & & & & \\
X & \longleftarrow & X \vee Y & \longrightarrow & Y & &
\end{array}$$

(for the second column, we have that homotopy coproducts are just strict coproducts for cofibrant objects (Proposition 3.2), but representable objects are cofibrant (Proposition 2.35)).

For the homotopy pushout of the third row of our first diagram, note that there is an elementary Nisnevich square:

$$\begin{array}{ccc}
(\mathbb{A}^n \setminus \{0\}) \times (\mathbb{A}^1 \setminus \{0\}) & \longrightarrow & \mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\}) \\
\downarrow & & \downarrow p \\
(\mathbb{A}^n \setminus \{0\}) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^{n+1} \setminus \{0\}
\end{array}$$

Indeed, it even forms a Zariski open cover. Therefore this square gives a homotopy pushout in $\mathrm{Spc}_{\mathbb{A}^1,*}$ by Proposition 2.38, as in the proof of Lemma 3.6. By construction, (iterated) projections from the product with $\mathbb{A}^1$ or $\mathbb{A}^n = (\mathbb{A}^1)^{\times n}$ are $\mathbb{A}^1$-weak equivalences, so $\mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\}$ and $(\mathbb{A}^n \setminus \{0\}) \times \mathbb{A}^1 \simeq \mathbb{A}^n \setminus \{0\}$, whence the conclusion. (In general, the homotopy pushout of a diagram like the third row is the *join* of the two spaces involved, which is equivalent to the suspension of the smash product).

We conclude that the homotopy pushouts of the two blue spans are the same. Indeed, homotopy colimits indexed by a product category $\mathcal{C} \times \mathcal{C}'$ can be computed first along $\mathcal{C}$ and then along $\mathcal{C}'$, or vice-versa: $\mathrm{hocolim}_{\mathcal{C}}\, \mathrm{hocolim}'_{\mathcal{C}} \cong \mathrm{hocolim}'_{\mathcal{C}}\, \mathrm{hocolim}_{\mathcal{C}}$ because $\mathrm{hocolim}_{\mathcal{C}}$ is a left Quillen functor (which exists because our category has all homotopy colimits) and hence it preserves homotopy colimits. Finally, we get that $\mathcal{S}^{2n+1,n+1}$ and $\mathbb{A}^{n+1} \setminus \{0\}$ are $\mathbb{A}^1$-weakly equivalent.

For the proof of (ii), note that the sphere $\mathcal{S}^{2n,n} \simeq \mathcal{S}^1 \wedge \mathcal{S}^{2n-1,n}$ is the homotopy cofiber of $\mathcal{S}^{2n-1,n} \to *$ by Lemma 3.7. By part (i) and by $\mathbb{A}^1$-homotopy invariance of homotopy cofibers, this is equivalently the homotopy cofiber of $\mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n$, namely the desired quotient.

Item (iii) is stated in [AE17], Corollary 4.46, as a consequence of the fact that every Nisnevich-locally trivial affine vector bundle $p : E \to X$ is an $\mathbb{A}^1$-equivalence (Proposition 4.44 in the same article).

The statement of (iv) appears in [ADF17], Theorem 2.2.5. The authors prove the statement when $S = \mathbb{Z}$ by induction, using the purity theorem (see Theorem 3.24), and then use a base change argument. $\square$

In particular, with the bigraded notation, $\mathbb{G}_m$ is the $(1,1)$-sphere and S^1 is the $(1,0)$-sphere. Here is a very rough heuristic to think about the bigrading, which I owe to William Hornslien during one of his talks ("Computing motivic homotopy types of families of hypersurfaces with polyhedral products"). One can look at the dimension as (real, Euclidean) spheres of the topological constructions corresponding to $\mathcal{S}^{m,n}$, either when we work over $S = \mathsf{Spec}(\mathbb{R})$ or over $\mathsf{Spec}(\mathbb{C})$. Then the first index would be the dimension in the complex case and the second index, the difference of the dimensions over $\mathbb{C}$ and $\mathbb{R}$. For instance, over $\mathbb{R}$, $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ is thought of as the real line minus one point, the latter is homotopy equivalent to $\mathbb{S}^0$. Whereas over $\mathbb{C}$, $\mathbb{A}^1 \setminus \{0\}$ is the complex plane minus the origin, topologically it is $\mathbb{S}^1$. Thus the indices should be $(1, 1-0) = (1,1)$. For S^1, it is the same simplicial set whether we work over $\mathbb{R}$ or $\mathbb{C}$ and it geometric realization is $\mathbb{S}^1$, so the indices should be $(1, 1-1) = (1,0)$. Similarly, for $\mathbb{A}^n \setminus \{0\}$, we think of it as $\mathbb{R}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ over $\mathbb{R}$ but over $\mathbb{C}$ it is $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R}^{2n} \setminus \{0\} \simeq \mathbb{S}^{2n-1}$. So it should be a motivic sphere with indices $(2n-1, 2n-1-(n-1)) = (2n-1, n)$. For $\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\})$, defined as the cofiber of $\mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n$, over $\mathbb{R}$ it looks like the cofiber of $\mathbb{S}^{n-1} \to \mathbb{R}^n \simeq \mathbb{D}^n$, i.e. $\mathbb{S}^n$. Over $\mathbb{C}$, it looks like the homotopy cofiber of $\mathbb{S}^{2n-1} \to \mathbb{C}^n \cong \mathbb{D}^{2n}$, i.e. $\mathbb{S}^{2n}$. So it should be a motivic sphere with indices $(2n, 2n-n) = (2n, n)$. This corresponds to the results of Proposition 3.9.

3.3 Invariance and the singular space functor

We will construct some motivic version $\mathsf{Sing}_{\mathbb{A}^1}$ of the singular set functor $\mathsf{Sing} : \mathsf{Top} \to {}_S\mathsf{Set}$, or maybe more precisely of the functor $|\mathsf{Sing}| : \mathsf{Top} \to \mathsf{Top}$. This functor has the property that the counit of the adjunction $|\bullet| \dashv \mathsf{Sing}$ is a weak equivalence $|\mathsf{Sing}(X)| \to X$ for all $X \in \mathsf{Top}$. We will have an analogue of this property in Theorem 3.16. According to [AE17], Sections 6.1 and 6.2, the $\mathsf{Sing}_{\mathbb{A}^1}$ construction also shares some features with the Quillen "plus" construction, which has the purpose of performing an abelianization of the fundamental group of a topological space without modifying homology (but higher homotopy groups might change a lot).

We first discuss further properties of $\mathbb{A}^1$-invariant simplicial presheaves and $\mathbb{A}^1$-rigid schemes (Definition 2.43).

Remark 3.10. Recall the functor $\mathsf{L}_{\mathbb{A}^1} := \mathsf{L}_{\mathcal{I}}$ from Definition 2.29, where $\mathcal{I}$ is the collection of (representatives for isomorphisms classes of) projections $U \times_S \mathbb{A}^1 \to U$ as U varies in Sm_S. Let X be an S-scheme (if X is not smooth over S, we ask for S to be Noetherian of finite Krull dimension).

Then X is $\mathbb{A}^1$-rigid if and only if $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X \simeq X$ as presheaves (in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$). Indeed, by Proposition 2.35, X is already Nisnevich fibrant. Thus $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X \simeq X$ in Spc if and only if X is $\mathbb{A}^1$-fibrant, i.e. if and only if X is $\mathbb{A}^1$-invariant by Remark 2.44.

Example 3.11 (Some $\mathbb{A}^1$-rigid objects (exercise 4.33 in [AE17])). If S is reduced, then $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid. If furthermore $S = \mathsf{Spec}(k)$ is a field, then any smooth projective curve of *positive genus* over k is $\mathbb{A}^1$-rigid. This is false for genus 0 curves: for instance $\mathbb{P}^1$ is not $\mathbb{A}^1$-rigid, by the same proof as Example 2.45, since there are morphisms of k-shemes $\mathbb{P}^1 \to \mathbb{P}^1$ not factoring through $\mathsf{Spec}(k)$.

Note that, in both cases, since the simplicial presheaves represented by these schemes are constant simplicial objects, we have to show that for all $U \in \mathsf{Sm}_S$, the projection induces a *bijection* $\mathsf{Sm}_S(U, \mathbb{G}_m) \to \mathsf{Sm}_S(U \times_S \mathbb{A}^1, \mathbb{G}_m)$ and similarly for the case of a curve. It is easy to see that this application is always injective: indeed it admits a retraction given by the zero section (because of contravariance) $U \times_S \iota_0 : U \times_S S \to U \times_S \mathbb{A}^1$ (since the post-composition of this section by the projection is just the identity on U). Thus in both cases we only have to show surjectivity.

The case of $\mathbb{G}_m$ over S reduced. We have to show that every morphism $f : U \times_S \mathbb{A}^1 \to \mathbb{G}_m$ factors through the projection. We work affine-locally: consider an open affine subset $\mathsf{Spec}(R)$ in S, by hypothesis R is a reduced ring. If $g : \mathbb{G}_m \to S$ and $u : U \to S$, $\tilde{u} : U \times_S \mathbb{A}^1 \to S$ are the structure maps, then $g^{-1}(\mathsf{Spec}(R)) = \mathsf{Spec}(R[t, t^{-1}])$ and $u^{-1}(\mathsf{Spec}(R)) =: \mathsf{Spec}(A)$ is affine, also $\tilde{u}^{-1}(\mathsf{Spec}(R)) \cong \mathsf{Spec}(A[x])$. Then $f : \mathsf{Spec}(A[x]) = f^{-1}(\mathsf{Spec}(R[t, t^{-1}])) \to \mathsf{Spec}(R[t, t^{-1}])$ is uniquely determined by an R-algebra homomorphism $f^\sharp : R[t, t^{-1}] \to A[x]$, i.e. it only depends on the image of t, which can only be a unit. Assume $f^\sharp(t) = \sum_{i\leq n} a_i x^i$ is a unit, then for every prime ideal $\mathfrak{p}$ in A, the reduction modulo $\mathfrak{p}$ of this polynomial is a unit in $(A/\mathfrak{p})[x]$ and thus it is constant since $A/\mathfrak{p}$ is a domain; so $a_i \in \mathfrak{p}$ for all $i > 0$. As $\mathfrak{p}$ varies over the primes of A we get $a_i \in \bigcap_{\mathfrak{p}\in \mathsf{Spec}(A)} \mathfrak{p} = \mathsf{Nil}(A) = \{0\}$ for all $i > 0$ (since we saw A was reduced). Thus $f^\sharp(t)$ is a constant polynomial, i.e. $f^\sharp$ factors through the inclusion $A \to A[x]$. This shows that the corresponding morphism of schemes factors through the projection, as desired. (Without reducedness, note that the non-constant polynomial $1 + \varepsilon x$ has inverse $1 - \varepsilon x$ in the ring $(A[\varepsilon]/(\varepsilon^2))[x]$ for instance.)

The case of a smooth projective curve X of positive genus over a field k. Once more we show that any morphism $f : U \times_S \mathbb{A}^1 \to X$ factors through the projection. The scheme $\mathbb{A}^1$ is an incomplete curve over k (i.e. it is an open subset of a smooth projective curve over k, namely $\mathbb{P}^1$). Then, it is a classical statement that every morphism $\mathbb{A}^1 \to X$ can be uniquely completed into a morphism $\mathbb{P}^1 \to X$ (see for instance [Vak17], 16.5.1). However, since $\mathbb{P}^1$ has genus 0, and X has positive genus, any such map must be constant (this is an easy consequence of the Riemann-Hurwitz formula, in the same spirit as exercise 21.7.E in [Vak17]). Therefore, intuitively, the morphism f "doesn't depend on the second coordinate".

We now make this precise. If we show that the composition

$$U \times_S \mathcal{A}^1 \xrightarrow{\pi} U \xrightarrow{\iota_0} U \times_S \mathcal{A}^1 \xrightarrow{f} X$$

is equal to f, we are done. We may assume that k is algebraically closed, because if the two morphisms agree on the based-changed schemes over $\bar{k}$, they agree for the original schemes ([Vak17], 9.2.I) (the base change of X is still a smooth projective curve). Then, we just have to check that the maps agree on the k-points ([Vak17], 10.2.B). We show a bit better, namely that for $\mathsf{Spec}(k) \to U$ a k-point, the composition above and f are equalized by $\mathsf{Spec}(k) \times_{\mathsf{Spec}(k)} \mathbb{A}^1 \to U \times_{\mathsf{Spec}(k)} \mathbb{A}^1$. By the argument above both compositions must be constant, and they are the same because in both cases the zero section $\mathsf{Spec}(k)$ is mapped to the image by f of the k-point under consideration.

Example 3.12 (Exercise 4.35 in [AE17])). Two $\mathbb{A}^1$-rigid *smooth* schemes over S are isomorphic in Sm_S if and only they are $\mathbb{A}^1$-equivalent in $\mathsf{Spc}_{\mathbb{A}^1}$. Indeed, we have seen in Remark 3.10 that $\mathbb{A}^1$-rigid smooth schemes are $\mathbb{A}^1$-fibrant (in particular, Nisnevich fibrant). But weak equivalences between fibrant objects in a Bousfield localization are exactly the weak equivalences in the model structure before localization (2.27). Therefore the presheaves represented by these schemes are weakly equivalent in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$. As constant simplicial objects, this happens if and only if they are isomorphic as presheaves of sets. By Yoneda lemma, this holds if and only if the schemes themselves are isomorphic.

Lemma 3.13 ($\mathbb{A}^1$-invariance criterion (exercise 4.26 in [AE17])). *If* $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$, *then* X *is* $\mathbb{A}^1$-*invariant if and only if for any* $U \in \mathsf{Sm}_S$ *the morphisms* $X(\iota_0), X(\iota_1) : X(U \times_S \mathbb{A}^1) \to X(U)$ *are homotopic (the maps* ι_0 *and* ι_1 *are the* 0- *and* 1-*section respectively, see in Definition 4.5 for their construction).*

Proof. Assume to begin with that X is $\mathbb{A}^1$-invariant. We temporarily adopt the "op" notation to keep track of the opposite categories (or contravariance). Then, for all $U \in \mathsf{Sm}_S$, if $\pi : U \times_S \mathbb{A}^1 \to U$ is the projection, by hypothesis we have weak equivalences of simplicial sets $X(\pi^{op}) : X(U) \to X(U \times_S \mathbb{A}^1)$ and therefore $\mathsf{map}(X(\pi^{op})^{op}, X(U)) : \mathsf{map}(X(U \times_S \mathbb{A}^1), X(U)) \to \mathsf{map}(X(U), X(U))$. In particular the latter induces a bijection on the connected components, but these are just the homotopy classes of simplicial maps. We conclude by noting that both $X(\iota_0^{op})$ and $X(\iota_1^{op})$ are sent to the class of the identity (since $\iota_j^{op} \circ \pi^{op} = (\pi \circ \iota_j)^{op} = \mathsf{id}$ for $j = 0, 1$), so they must be homotopic.

Conversely, assume that $X(\iota_0^{op})$ and $X(\iota_1^{op})$ are homotopic for any choice of smooth scheme U. Fix $U \in \mathsf{Sm}_S$, we want to show that the map of simplicial sets $X(\pi^{op}) : X(U) \to X(U \times_S \mathbb{A}^1)$ is a (simplicial) homotopy equivalence, then it will in particular be a weak equivalence. We claim that $X(\iota_0^{op})$ is a homotopy inverse for $X(\pi^{op})$. We already have $X(\iota_0^{op}) \circ X(\pi^{op}) = X((\pi \circ \iota_0)^{op}) = X(\mathsf{id}_U^{op})$. We now show that the other composition $X(\pi^{op}) \circ X(\iota_0^{op})$ is homotopic to the identity on $X(U \times_S \mathbb{A}^1)$. Applying our hypothesis to the smooth scheme $\widetilde{U} := U \times_S \mathbb{A}^1$, we have a simplicial homotopy H between the maps $X(\widetilde{\iota}_j^{op}) : X((U \times_S \mathbb{A}^1) \times_S \mathbb{A}^1) \to X((U \times_S \mathbb{A}^1))$ for $j = 0, 1$ where $\widetilde{\iota}_j$ denotes the j-section $\widetilde{U} \to \widetilde{U} \times_S \mathbb{A}^1$. Consider the composition $\widetilde{H}$ given by:

$$X(\widetilde{U}) \times \Delta^1 \xrightarrow{X(\mu^{op}) \times \mathsf{id}_{\Delta^1}} X(\widetilde{U} \times_S \mathbb{A}^1) \times \Delta^1 \xrightarrow{\quad H \quad} X(U \times_S \mathbb{A}^1)$$

where $\mu : \widetilde{U} \times_S \mathbb{A}^1 = U \times_S \mathbb{A}^1 \times_S \mathbb{A}^1 \to U \times_S \mathbb{A}^1 = \widetilde{U}$ is given by the identity on U and the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, induced by the ring map $\mathbb{Z}[t] \to \mathbb{Z}[x,y]$, $t \mapsto xy$ (based changed by S). We check that $\widetilde{H}$ is a simplicial homotopy between $X(\pi^{op}) \circ X(\iota_0^{op})$ and the identity. The precomposition of $\widetilde{H}$ by the inclusion ∂_j^* in the j-th vertex of Δ^1 is equal to:

$$H \circ (X(\mu^{op}) \times \mathrm{id}_{\Delta^1}) \circ \partial_j^* = H \circ \partial_j^* \circ X(\mu^{op}) = X(\widetilde{\iota_j}^{op}) \circ X(\mu^{op}) = X((\mu \circ \widetilde{\iota_j})^{op})$$

and $\mu \circ \widetilde{\iota_j} : U \times_S \mathbb{A}^1 \to U \times_S \mathbb{A}^1 \times_S \mathbb{A}^1 \to U \times_S \mathbb{A}^1$ is induced by the map $\mathbb{Z}[t] \to \mathbb{Z}[x,y] \to Z[t]$, mapping $t \mapsto xy \mapsto jt$, therefore for $j = 1$ we get the identity and for $j = 0$, the map corresponding to $\iota_0 \circ \pi$, as desired. □

After this parenthesis, we are ready to build the advertised $\mathrm{Sing}_{\mathbb{A}^1}$ functor and discuss its properties. For a topological space X and $n \in \mathbb{N}$, the n-simplices of $\mathrm{Sing}(X)$ are given by all continuous maps from the standard topological n-simplex Δ^n to X. Recall that Δ^n can be described as

$$\Delta^n := \left\{ (x_0, \dots x_n) \in \mathbb{R}^{n+1}_{\geq 0} \,\middle|\, \sum_{i \leq n} x_i - 1 = 0 \right\}.$$

In our setting:

Definition 3.14 (Singular space functor ([AE17], Definition 4.23))**.** Let $\Delta^\bullet$ be the cosimplicial scheme defined by $\Delta^n = S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[x_0, \dots, x_n]/(x_0 + \dots + x_n - 1))$ (the context should be enough to distinguish it from the standard simplex $\Delta^n \in {}_S\mathrm{Set}$; the simplicial constant presheaf equal to this simplicial set was denoted $\underline{\Delta^n}$).

The *singular space functor* $\mathrm{Sing}_{\mathbb{A}^1} : {}_S\mathrm{Pre}(\mathrm{Sm}_S) \to {}_S\mathrm{Pre}(\mathrm{Sm}_S)$ associates to $X \in {}_S\mathrm{Pre}(\mathrm{Sm}_S)$ the simplicial presheaf $\mathrm{Sing}_{\mathbb{A}^1}(X) = |X(- \times_S \Delta^\bullet)|$, where realization is taken along the indices represented by the bullet • (X being contravariant, we indeed obtain a simplicial and not a cosimplicial object).

Alternatively, [MV99] define $\mathrm{Sing}_{\mathbb{A}^1}(X)$ as the diagonal of the bisimplicial presheaf $X(- \times \Delta^\bullet)$ (see p 88 and use representability). These two definitions are isomorphic: to see this, use (objectwise) the fact that the geometric realization of a bisimplicial set along the first index, respectively second index, is isomorphic to its diagonal (IV.1.4 and beginning of VII.3 in [GJ09]).

Although Δ^n is isomorphic as a scheme to $\mathbb{A}^n$, using this model allows us to define the cofaces and codegeneracies of $\Delta^\bullet$ as in the simplicial/topological setting.

The singular functor comes with a natural map $X \to \mathrm{Sing}_{\mathbb{A}^1}(X)$ for any simplicial presheaf X, with $X_n(U) \to |X(U \times \Delta^\bullet)_n|$ induced for all $n \in \mathbb{N}$ and $U \in \mathrm{Sm}_S$ at level 0, identifying $X(U)$ with $X(U \times \Delta^0)$ (it is an inclusion).

Remark 3.15. Let $X \in \mathrm{Sm}_S$ be a representable presheaf. Then $\mathrm{Sing}_{\mathbb{A}^1}(X)(S) \in {}_S\mathrm{Set}$ looks very much like the construction $|\mathrm{Sing}(X')|$ for X' a topological space. Indeed, its n-simplices are given by $|\mathrm{Sm}_S(S \times_S \Delta^\bullet, X)| = |\mathrm{Sm}_S(\Delta^\bullet, X)|$.

The singular functor is a motivic version of anterior constructions in other contexts, which have proven to be useful and well-behaved. Morel and Voevosky introduce it in the general case of a *site with interval* (the interval being $\mathbb{A}^1$ here) for "computational" reasons (see [MV99], p 87); in particular they use it to prove left-properness of their model structure. They require (and prove) the following conditions for this construction:

(i) There is a natural transformation $\mathrm{id} \to \mathrm{Sing}_{\mathbb{A}^1}$ with components monomorphisms and weak equivalences in $\mathrm{Spc}_{\mathbb{A}^1}$.
(ii) The functor $\mathrm{Sing}_{\mathbb{A}^1}$ takes $* \to \mathbb{A}^1$ to a weak equivalence.
(iii) The functor $\mathrm{Sing}_{\mathbb{A}^1}$ preserves fibrations in $\mathrm{Spc}_{\mathbb{A}^1}$.

We have seen the existence of this natural transformation with monomorphisms as components just above. The fact that the components are weak equivalences and part (ii) follow from Theorem 3.16 below. Item (iii) is proved thanks to the existence of a left adjoint that preserves cofibrations and weak equivalences (Corollary 3.13 p 91 in [MV99], we will come back to it to prove Theorem 4.4).

Theorem 3.16 (Properties of $\mathrm{Sing}_{\mathbb{A}^1}$ ([AE17], Theorem 4.25))**.** *Assuming that S is separated and Noetherian. For all* $X \in {}_S\mathrm{Pre}(\mathrm{Sm}_S)$*:*

(i) $\mathsf{Sing}_{\mathbb{A}^1}(X)$ *is* $\mathbb{A}^1$*-invariant*

(ii) *The natural map* $X \to \mathsf{Sing}_{\mathbb{A}^1}(X)$ *is an* $\mathbb{A}^1$*-equivalence*

Proof. To prove (i), we use the criterion of Lemma 3.13. Let $U \in \mathsf{Sm}_S$. We have to exhibit a simplicial homotopy between $(\mathsf{Sing}_{\mathbb{A}^1}(X))(\iota_0), (\mathsf{Sing}_{\mathbb{A}^1}(X))(\iota_1) : (\mathsf{Sing}_{\mathbb{A}^1}(X))(U \times_S \mathbb{A}^1) \to (\mathsf{Sing}_{\mathbb{A}^1}(X))(U)$ the maps induced by the 0- and 1-sections. Using the remark below Definition 3.14, we choose to consider the singular functor as realization along the other index as in our original definition. To define a simplicial homotopy we need maps

$$h_i : \underbrace{(\mathsf{Sing}_{\mathbb{A}^1}(X))(U \times_S \mathbb{A}^1)_n}_{|X_\bullet(U\times_S \mathbb{A}^1 \times_S \Delta^n)|} \to \underbrace{(\mathsf{Sing}_{\mathbb{A}^1}(X))(U)_{n+1}}_{|X_\bullet(U\times_S \Delta^n)|}$$

for all $0 \leq i \leq n$ and $n \geq 0$ satisfying certain conditions with respect to the faces and degeneracies (described in [Wei94], 8.3.11).

There are morphisms of schemes $\theta_i : \Delta^{n+1} \to \mathbb{A}^1 \times_S \Delta^n$ reproducing the usual simplicial (or topological) subdivision of $\Delta^1 \times \Delta^n \in {}_s\mathsf{Set}$ into copies of Δ^{n+1}. For all vertices of Δ^{n+1} with index $j \leq i$, the map is induced by the 0-section and the inclusion on the j-th vertex of Δ^n. For $j > i$, the map is induced by the 1-section and the inclusion on the $(j-1)$-th vertex of Δ^n. Choose h_i to be the realization of $X(\theta_i \times_S \mathrm{id}_U)$. Checking all the requirement described in [Wei94] is easy but a bit tedious, so we only check some of them.

- $\partial_0 \circ h_0 = (\mathsf{Sing}_{\mathbb{A}^1}(X))(\iota_1)$: by functoriality of X we only have to show that the composition $U \times \Delta^n \xrightarrow{\partial_0} U \times_S \Delta^{n+1} \xrightarrow{\theta_0} U \times_S \mathbb{A}^1 \times_S \Delta^n$ is the 1-section. The first map sends a vertex v_j to v_{j+1} and then the second map sends it to v_j in the 1-section, as desired.
- $\partial_{n+1} \circ h_n = (\mathsf{Sing}_{\mathbb{A}^1}(X))(\iota_0)$: in a similar way, it suffices to compute the composition $U \times \Delta^n \xrightarrow{\partial_{n+1}} U \times_S \Delta^{n+1} \xrightarrow{\theta_n} U \times_S \mathbb{A}^1 \times_S \Delta^n$, the first map sends a vertex v_j to v_j itself and then the section map sends it to v_j in the 0-section, as we needed.

For part (ii), we want to show that for any $\mathbb{A}^1$-local object $Y \in \mathsf{Spc}_{\mathbb{A}^1}$, there is a weak equivalence of simplicial sets $\mathsf{map}(\mathsf{Sing}_{\mathbb{A}^1}(X), Y) \to \mathsf{map}(X, Y)$. We have seen above that $\mathsf{Sing}_{\mathbb{A}^1}(X)$ could be expressed as the realization of the bisimplicial presheaf $X(- \times_S \Delta^\bullet)$. We will now show that $\mathsf{map}(X(- \times_S \Delta^n), Y) \to \mathsf{map}(X, Y)$ is a weak equivalence of simplicial sets, and this will suffice to prove the claim. Indeed, if this holds, we have weak equivalences $\mathsf{map}(X, Y) \simeq \mathsf{holim}_{[n]\in\Delta} \mathsf{map}(X_n, Y)$ and $\mathsf{map}(\mathsf{Sing}_{\mathbb{A}^1}(X), Y) \simeq \mathsf{holim}_{[n]\in\Delta} \mathsf{map}(\mathsf{Sing}_{\mathbb{A}^1}(X)_n, Y)$, where X_n and $\mathsf{Sing}_{\mathbb{A}^1}(X)_n$ denote the corresponding constant simplicial presheaves, as in the proof of Lemma 2.32 (the claim that $X \simeq \mathsf{hocolim}_{[n]\in\Delta^{op}} X_n$ also holds in $\mathsf{Spc}_{\mathbb{A}^1}$, because the identity ${}_s\mathsf{Pre}(\mathsf{Sm}_S) \to \mathsf{Spc}_{\mathbb{A}^1}$ is left-Quillen). In particular it suffices to show that $\mathsf{map}(\mathsf{Sing}_{\mathbb{A}^1}(X)_n, Y) \to \mathsf{map}(X_n, Y)$ is a weak equivalence of simplicial sets for all $n \in \mathbb{N}$. We have $\mathsf{Sing}_{\mathbb{A}^1}(X)_n = X_n(- \times \Delta^n)$ (choosing the interpretation of the realization as the diagonal) and thus what we need follows from our assumption applied to the constant simplicial presheaf X_n.

We now prove the claim. By definition of $\mathbb{A}^1$-local weak equivalences, it suffices to show that $X \to X(- \times_S \Delta^n)$ is an $\mathbb{A}^1$-equivalence for all $n \in \mathbb{N}$. We will use the criterion of Lemma 4.6: it suffices to show that this map is an $\mathbb{A}^1$ homotopy equivalence. We reduce to the case $n = 1$ as follows: we have $X(- \times_S \Delta^n) \cong X(- \times_S \Delta^{n-1} \times_S \Delta^1)$ (since we have isomorphisms of schemes $\Delta^1 \times_S \Delta^{n-1} \cong \mathbb{A}^1 \times_S \mathbb{A}^{n-1} \cong \mathbb{A}^n \cong \Delta^n$). Thus if the case $n = 1$ holds true, applying it to the simplicial presheaf $X(- \times_S \Delta^{n-1})$ shows that $X(- \times_S \Delta^{n-1}) \to X(- \times_S \Delta^n)$ (induced by projection on a face) is an $\mathbb{A}^1$-equivalence. We can repeat this argument inductively.

The map $X(\pi) : X \to X(- \times_S \Delta^1)$ we have to consider is induced by the projection. Our candidate for a homotopy inverse is the map $X(\iota_0)$ induced by the 0-section (using the isomorphism of schemes $\Delta^1 \cong \mathbb{A}^1$). Thus $X(\iota_0) \circ X(\pi) = X(\pi \circ \iota_0)$ is the identity. We want to find an $\mathbb{A}^1$-homotopy between the other composite and the identity of $X(- \times_S \Delta^1)$. This means that we have to define a map $H : X(- \times_S \Delta^1) \times \mathbb{A}^1 \cong X(- \times_S \Delta^1) \times \Delta^1 \to X(- \times_S \Delta^1)$. Using objectwise the usual mapping space adjunction, this is precisely the data of an adjoint map $H' : X(- \times_S \Delta^1) \to X(- \times_S \Delta^1 \times_S \Delta^1)$. This is obtained exactly as in the proof of Lemma 3.13, using $\Delta^1 \cong \mathbb{A}^1$ and the multiplication map $\mathbb{A}^1 \times_S \mathbb{A}^1 \to \mathbb{A}^1$, induced on the rings by the morphism $\mathbb{Z}[t] \to \mathbb{Z}[x, y]$, $t \mapsto xy$. □

The singular construction allows us to describe more easily the $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$ of Remark 2.28. Precisely, we have the following theorem:

Theorem 3.17 (Description of the $\mathbb{A}^1$-localization (see for example [AE17], Theorem 4.27)). *Assume that* S *is Noetherian of finite Krull dimension. The functor* $L_{\mathbb{A}^1}L_{Nis} : {}_S\mathsf{Pre}(\mathsf{Sm}_S) \to \mathsf{Spc}_{\mathbb{A}^1}$ *is equivalent to the countable iteration* $(L_{Nis}\mathsf{Sing}_{\mathbb{A}^1})^{\circ\mathbb{N}}$*, namely for all* $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$*, there is a natural equivalence in* ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ *between* $\mathrm{hocolim}_n(L_{Nis}\mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X)$ *and* $L_{\mathbb{A}^1}L_{Nis}(X)$*. This sequential homotopy colimit is both a homotopy colimit in* ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ *and* Spc.

Remark 3.18. The assumptions on S ensure that a countable number of iterations is sufficient (see [Mor12], Remark 6.21)

3.4 Thom spaces

Given a topological vector bundle $\gamma : E \to X$, one can define the *Thom space* $\mathsf{Th}(\gamma)$ of this bundle in several ways. One (perhaps a bit sloppy) definition would be to say that is is the topological space obtained by taking the one-point compactification of each fiber individually (fibers are real vector spaces), and then collapsing the subspace consisting of all the points at infinity that were added. One can also define $\mathsf{Th}(\gamma)$ as the collapse of the unit disk bundle (picking the unit disk in each fiber) by the unit sphere bundle (picking the unit sphere in each fiber). This requires a compatible choice of inner products on the fibers, but it is possible because a vector bundle is locally trivial by definition.

To perform the same construction for schemes, recall that vector bundles are defined as follows in algebraic geometry (see for example the Stacks project, Tag 01M1):

Definition 3.19 (Algebraic vector bundle). A morphism $\nu : E \to X$ of schemes if called an *(algebraic) vector bundle* if it is an affine morphism such that $\nu_*(\mathcal{O}_E)$ is endowed with a graded $\mathcal{O}_X$-algebra structure, making it isomorphic to $\mathsf{Sym}(\nu_*(\mathcal{O}_E)_1)$.

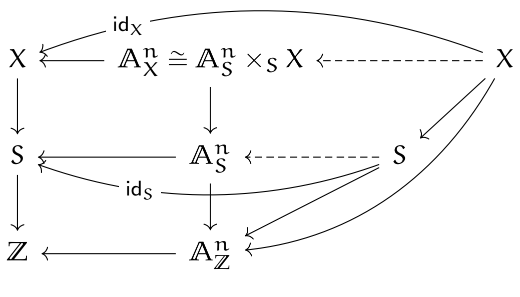
Then the category of algebraic vector bundles over X is anti-equivalent to the category of quasi-coherent sheaves over X.

Recall that the motivic sphere $\mathcal{S}^{2n,n}$ was $\mathbb{A}^1$-equivalent to $\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\})$ (Proposition 3.9). If $S = \mathsf{Spec}(\mathbb{R})$, for the topological intuition $\mathbb{A}^n$ plays the role of $\mathbb{R}^n \simeq \mathbb{D}^n$ and then $\mathbb{A}^n \setminus \{0\}$ is viewed as $\mathbb{D}^n \setminus \{0\} \simeq S^{n-1}$. We then generalize the definition of the topological Thom space as a quotient of the unit disk bundle by the unit sphere bundle as follows:

Definition 3.20 (Thom space of an algebraic vector bundle). If $\nu : E \to X$ is a vector bundle in Sm_S, the *Thom space of* ν is defined as $\mathsf{Th}(\nu) := E/(E - X)$ where X embeds into E as the zero section.

Example 3.21. If $E \to X$ is a trivial (algebraic) vector bundle of rank n, then there is an $\mathbb{A}^1$-equivalence $\mathsf{Th}(E) \simeq (\mathbb{P}^1)^{\wedge n} \wedge X_+$.

Indeed, up to isomorphism, by definition of a trivial bundle, we may assume that $E \to X$ is just the projection $\mathbb{A}^n_X \cong \mathbb{A}^n_S \times_S X \to X$ (since $\mathsf{Sym}(\mathcal{O}_X^{\oplus n})$ just gives the trivial affine bundle of rank n over X). Then, by definition, the Thom space of this bundle is computed as the homotopy cofiber $* \longleftarrow (\mathbb{A}^n \times_S X) \setminus X \longrightarrow \mathbb{A}^n \times_S X$, or equivalently $* \longleftarrow ((\mathbb{A}^n \setminus S) \times_S X)_+ \longrightarrow (\mathbb{A}^n \times_S X)_+$. Indeed, note that $(\mathbb{A}^n \setminus S) \times_S X = (\mathbb{A}^n \times_S X) \setminus X$, because the zero section $X \to \mathbb{A}^n_X \cong \mathbb{A}^n_S \times_S X$ is exactly the base change by the zero section $S \to \mathbb{A}^n_S$ of the projection $\mathbb{A}^n_S \times_S X \to \mathbb{A}^n_S$. This holds because we have a diagram as follows:



and we use the reverse pasting law for pullbacks in the top trapezium with corners X, X, S, and S, to deduce that the square at its right-hand side is a pullback.

Now we use our favorite trick to compute homotopy pushout: we have a commutative diagram:

$$
\begin{array}{ccccccc}
* & \longleftarrow & * \vee X_+ \cong X_+ & = & * \times X_+ \cong X_+ & \qquad & * \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
X_+ & \longleftarrow & (\mathbb{A}^n_S \setminus S) \vee X_+ & \longrightarrow & (\mathbb{A}^n_S \setminus S) \times X_+ & & ((\mathbb{A}^n_S \setminus S) \times_S X)_+ \\
\| & & \downarrow & & \downarrow & & \downarrow \\
X_+ & \longleftarrow & \mathbb{A}^n_S \vee X_+ & \longrightarrow & \mathbb{A}^n_S \times X_+ & & (\mathbb{A}^n_S \times_S X)_+ \\
 & & & & & & \\
* & \longleftarrow & (\mathbb{P}^1_S)^{\wedge n} \vee X_+ & \longrightarrow & (\mathbb{P}^1_S)^{\wedge n} \times_S X_+ & &
\end{array}
$$

where each row and column is a homotopy pushout, by Proposition 3.9. For the pushout of the third row, we can just replace $\mathbb{A}^n_S$ by the point, up to $\mathbb{A}^1$-equivalence, to do the computation. Or we can use again a computation with four commutative squares, which works to compute the homotopy pushout for the second row of the above diagram too (where $Y = \mathbb{A}^n_S$ or $Y = \mathbb{A}^n_S \setminus S$):

$$
\begin{array}{ccccccc}
* & \longleftarrow & \emptyset & \longrightarrow & X & \qquad & X_+ \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y & \longleftarrow & \emptyset & \longrightarrow & X & & Y \vee (X_+) \\
\| & & \downarrow & & \downarrow & & \downarrow \\
Y \times * & \longleftarrow & \emptyset & \longrightarrow & Y \times X & & Y \times X_+ \\
 & & & & & & \\
* & \longleftarrow & \emptyset & \longrightarrow & Y \times X & &
\end{array}
$$

We saw that the homotopy pushout of the vertical blue space in the first diagram above was the desired Thom space, and so it is equivalent to the homotopy pushout of the horizontal blue span, namely $(\mathbb{P}^1_S)^{\wedge n} \wedge X_+$.

Thom spaces are involved in the purity theorem. "Purity" refers to a general concept, in a nutshell: in any cohomology theory, for suitable closed immersions $Z \to X$, there should exist a "purity isomorphism" $H^{r-2c}(Z) \xrightarrow{\sim} H^r_Z(X)$ where H_Z denotes cohomology supported in Z. This isomorphism induces a *Gysin long exact sequence*

$$\cdots \to H^{r-1}(X \setminus Z) \to H^{r-2c}(Z) \to H^r(X) \to H^r(X \setminus Z) \to \ldots$$

For example:

Theorem 3.22 (Purity theorem in étale cohomology (Thomason and Gabber, see [Fuj02])). *Let k be an algebraically closed field, and $c \in \mathbb{N}^*$, such that $\mathsf{char}(k)$ is coprime to c. Let $Z \to X$ be a regular closed immersion of varieties over k, and assume that Z is of pure codimension c in X (all its irreducible components have codimension c). Then for any locally constant sheaf $\mathcal{F}$ of $(\mathbb{Z}/n\mathbb{Z})$-modules for some $n \in \mathbb{N}^*$, there is a canonical* purity isomorphism *of cohomology groups:*

$$H^{r-2c}_{\acute{e}t}(Z, \mathcal{F}(-c)) \longrightarrow H^r_Z(X, \mathcal{F})$$

where H_Z denotes cohomology with support in Z.

Theorem 3.23 (Purity theorem in topology (stated and proof sketched in [AE17])). *Let $Z \to X$ be a closed immersion of smooth manifolds of real codimension c. Then, if ν_Z is the normal bundle, there is a homotopy equivalence $\mathsf{Th}(\nu_Z) \simeq X/(X - Z)$, and for every field k, there is a canonical isomorphism of cohomology groups:*

$$\tilde{H}^{i-c}_{\mathsf{sing}}(Z, k) \longrightarrow \tilde{H}^i_{\mathsf{sing}}(\mathsf{Th}(\nu_Z), k)$$

Now, with the construction of an $\mathbb{A}^1$-homotopy theory we studied in Section 2, an analog of this statement holds:

Theorem 3.24 (Purity theorem in $\mathbb{A}^1$-homotopy (Section 3, Theorem 2.23 in [MV99]). *Let* $Z \to X$ *be a closed embedding in* Sm_S *and* ν_Z *be the normal bundle. There is an* $\mathbb{A}^1$*-weak equivalence* $X/(X-Z) \to \mathsf{Th}(\nu_Z)$*, natural in* $\mathsf{Ho}(\mathsf{Spc}_{\mathbb{A}^1})$ *for smooth pairs (i.e. pairs* (X,Z) *as above, where a morphism* $(X,Z) \to (X',Z')$ *of smooth pairs is a pullback square*

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & X' \end{array}$$

in Sm_S*).*

Proof. We give a quick summary of the proof of [AE17]. A very interesting and detailed outline with illustrations can be found in [WW20], Theorem 2.10.

- The idea is to obtain a sequence of natural $\mathbb{A}^1$-equivalences:

  $$X/X - Z \simeq D_ZX/D_ZX - (Z \times_S \mathbb{A}^1) \simeq N_ZX/N_ZX - Z = \mathsf{Th}(\nu_Z)$$

  where $\nu_Z : N_ZX \to Z$ is the normal bundle, and

  $$D_ZX = \mathsf{Bl}_{Z\times_S\{0\}}(X \times_S \mathbb{A}^1) - \mathsf{Bl}_{Z\times_S\{0\}}(X \times_S \{0\})$$

  where Bl denotes the blow-up. The bundle D_ZX is the "deformation to the normal bundle", in the sense that fibers of $Z \times_S \mathbb{A}^1 \to D_ZX$ over $\mathbb{A}^1$ vary from the immersion $Z \hookrightarrow X$ to the zero section of the normal bundle N_ZX.
- Define a weakly excisive morphism of smooth pairs to be a morphism of smooth pairs that induces equivalences as above. So the theorem amount to show that $(X,Z) \to (D_ZX, Z \times_S \mathbb{A}^1)$ and $(N_ZX, Z) \to (D_ZX, Z \times_S \mathbb{A}^1)$ are weakly excisive.
- This property holding true is viewed as a property of the pair (X,Z). Then, one checks this property on smaller "generating data", so that in the end it must hold for all smooth pairs. More precisely, one has to check that the property:
  - holds for the zero sections (the pairs $(\mathbb{A}^n_Z, Z)$),
  - can be checked Zariski locally on a smooth pair,
  - transports and pulls back along *Nisnevich morphisms* of smooth pairs (namely morphisms $f : (X,Z) \to (X',Z')$ such that $X \to X'$ is étale and $f^{-1}(Z') \to Z'$ an isomorphism).

This makes use of the Nisnevich topology in the fact that Nisnevich morphisms are very reminiscent of the properties asked to define a Nisnevich square (in fact, given a Nisnevich morphism as above, $X \to X'$ and $X' \setminus Z' \to X'$ define an elementary Nisnevich distinguished square). □

The topological and motivic versions rely on the Thom space of the normal bundle, but the statement for the Thom space can be translated into the existence (and naturality) of a purity isomorphism.

4 Homotopy for motivic spaces

This section introduces the analog of homotopy groups for motivic spaces, and related constructions, such as classifying spaces and Eilenberg-MacLane objects associated with a (Nisnevich) sheaf of groups. These objects will appear in our discussion of Postnikov towers in Section 5.

4.1 Homotopy sheaves

The "homotopy groups" of motivic spaces will be a bit different from their topological counterparts, in that they will themselves be sheaves on Sm_S. In particular, they can be viewed as objects of Spc or $\mathsf{Spc}_{\mathbb{A}^1}$. Since we have three stages in our construction, namely ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$, Spc and $\mathsf{Spc}_{\mathbb{A}^1}$, we have three different homotopy theories. And thus, three different kinds of homotopy sheaves:

Definition 4.1 (Nisnevich and $\mathbb{A}^1$ homotopy sheaves)**.** Let $n \in \mathbb{N}$.

- The n*-th simplicial homotopy sheaf* $\pi_n(X)$ of $X \in {}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)_*$ is the Nisnevich sheafification (see 2.7) of the *homotopy presheaf* of groups on Sm_S given by $U \mapsto \pi_n(X(U))$, where $\pi_n(X(U))$ is the homotopy group for simplicial sets (the simplicial set $X(U)$ is pointed since X is).
- The n*-th Nisnevich homotopy sheaf* $\pi_n^{\mathsf{Nis}}(X)$ of $X \in \mathsf{Spc}_*$ is the Nisnevich sheafification of the *homotopy presheaf* of groups (see Remark 4.2 below) on Sm_S given by $U \mapsto \mathsf{Ho}(\mathsf{Spc}_*)(\mathcal{S}^n \wedge U_+, X)$, where U_+ is U with a disjoint basepoint added, viewed as an object of ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)_*$.
- The n*-th* $\mathbb{A}^1$*-homotopy sheaf* $\pi_n^{\mathbb{A}^1}(X)$ of $X \in \mathsf{Spc}_{\mathbb{A}^1,*}$ is the Nisnevich sheafification of the presheaf on Sm_S given by $U \mapsto \mathsf{Ho}\left(\mathsf{Spc}_{\mathbb{A}^1,*}\right)(\mathcal{S}^n \wedge U_+, X)$.

The hom-sets in the homotopy categories $\mathsf{Ho}(\mathsf{Spc}_*)$ and $\mathsf{Ho}\left(\mathsf{Spc}_{\mathbb{A}^1,*}\right)$ as above are denoted by $[\mathcal{S}^n \wedge U_+, X]_{\mathsf{Nis},*}$ and $[\mathcal{S}^n \wedge U_+, X]_{\mathbb{A}^1,*}$ respectively (and similarly for hom-sets in the homotopy categories of ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)_*$ or ${}_{\mathsf{S}}\mathsf{Set}_*$, and their non-pointed analogs).

Homotopy sheaves keep track of the "Nisnevich- or $\mathbb{A}^1$- homotopy classes of maps" between the n-th simplicial sphere and the sections $X(U)$ of X, as U varies in Sm_S. In topology, homotopy groups are closely related with the loop space construction, which is an adjoint to suspension: indeed, for X a pointed topological space, we have $\pi_{n+1}(X) \cong \pi_n(\Omega X) \cong \pi_1(\Omega^n X)$ for all $n \in \mathbb{N}^*$. We have the same kind of adjunction in our setting by subsection 3.1. In our case, since suspension in the categorical sense corresponds to taking the smash product with the *simplicial circle*, the homotopy sheaves should therefore also use the simplicial circle and spheres (rather than the Tate circle or other bigraded spheres). Moreover, using the simplicial spheres allows us to make the homotopy sheaves into sheaves of groups:

Remark 4.2 (Group structure on the homotopy sheaves). We show that the homotopy presheaves we defined above can be endowed with the structure of presheaves of groups, and this induces by sheafification a structure of sheaves of groups on the homotopy sheaves.

Let $n \geq 1$. We have to endow $[S^n \wedge U_+, X]_{\mathbb{A}^1,*}$ with a group structure for any $X \in {}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)_*$ and $U \in \mathsf{Sm}_S$, natural in U and X. Recall the functor $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}$ from Definition 2.29 (composition of $\mathsf{L}_{\mathcal{I}}$ and $\mathsf{L}_{\mathcal{J}}$ with $\mathcal{I}$ given by (isomorphism classes of) projections $U \times_S \mathbb{A}^1 \to U$ as U varies in Sm_S, and $\mathcal{J}$ the (isomorphism classes of) Nisnevich hypercovers). Since X is naturally $\mathbb{A}^1$-equivalent to $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)$, we have a natural bijection

$$[\mathcal{S}^n \wedge U_+, X]_{\mathsf{A}^1,*} \cong [\mathcal{S}^n \wedge U_+, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)]_{\mathbb{A}^1,*}$$

Note that $\mathcal{S}^n \wedge U_+ \in \mathsf{Spc}_{\mathbb{A}^1,*}$ is cofibrant. Indeed, both representable objects and $\mathcal{S}^n$ are cofibrant in $\mathsf{Spc}_{\mathbb{A}^1}$, and the smash product of two cofibrations is a cofibration by the pushout product axiom. Also, $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)$ is by definition fibrant. Therefore, using the derived adjunction of the Quillen pair of Proposition 3.3, we can compute:

$$\begin{aligned}
[\mathcal{S}^n \wedge U_+, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)]_{\mathbb{A}^1,*} &\cong [\mathcal{S}^n \wedge U_+, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)]_{{}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S),*} \\
&\cong [\mathcal{S}^n, \mathsf{map}_{{}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S),*}(U_+, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X))]_{{}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S),*} \\
&\cong [\mathcal{S}^n, \mathsf{map}^{o}_{{}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S),*}(U_+, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X))]_{{}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S),*} \\
&\cong [S^n, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)(U)]_{{}_{\mathsf{S}}\mathsf{Set},*} \qquad (\star)
\end{aligned}$$

To check $(\star)$, note that a (pointed) map from a simplicial constant presheaf $\underline{A}$ (for A a (pointed) simplicial set) to $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ corresponds uniquely to a (pointed) map $A \to X(S)$ of simplicial sets (because S is terminal in Sm_S). Therefore, maps $\mathcal{S}^n \to \mathsf{map}^o_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)}(U, X)$ corresponds to maps $S^n \to \mathsf{map}(U(S), X(S))$. We claim that $\mathsf{map}(U(S), X(S)) = \mathsf{map}(U, X)$; to show this we compare the m-simplices. We have, for X fibrant:

$$\begin{aligned}
\mathsf{map}(U, X)_m &= {}_S\mathsf{Pre}(\mathsf{Sm}_S)(U \times \underline{\Delta^m}, X) \\
&\cong {}_S\mathsf{Pre}(\mathsf{Sm}_S)(\underline{\Delta^m}, \mathsf{map}_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)}(U, X)) && \text{(by adjunction)} \\
&\cong {}_S\mathsf{Set}(\Delta^m, \mathsf{map}^o_{{}_S\mathsf{Pre}(\mathsf{Sm}_S)}(U, X)(S)) && (\underline{\Delta^m} \text{ is a simplicial constant presheaf}) \\
&\cong {}_S\mathsf{Set}(\Delta^m, \mathsf{map}(U(S), X(S))) \\
&\cong \mathsf{map}(U(S), X(S))_m
\end{aligned}$$

By fibrancy, $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)$ takes its values in Kan complexes, and thus $[S^n, \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)(U)]_{{}_S\mathsf{Set},*}$ is the set of (pointed) homotopy classes of maps of simplicial sets $S^n \to \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)(U)$, which is endowed with a group structure : it is the underlying set of the n-th homotopy group of the (pointed) fibrant simplicial set $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X)(U)$ (for the definition of the latter, we refer to section 1.4 in [May92]. Addition of simplices is defined using the horn-filling condition: the simplices to be added are considered as well chosen faces of some horn, we pick an extension, and the face that was added represents the result of the operation).

In the case of motivic *spectra* (see subsection 6.3), one defines bigraded (stable) homotopy sheaves of a (pointed) motivic spectrum X by using the hom-sets in the motivic stable homotopy category $[\mathcal{S}^{p,q}, X]$ with source the bigraded spheres $\mathcal{S}^{p,q}$ (viewed as motivic spectra). This time, there is a group structure on $[\mathcal{S}^{p,q}, X] \cong [\mathcal{S}^{p,q}, \Omega\Sigma(X)]$ because the target is a loop space (in topology, we would talk about an H-group). Indeed, we are now working with a stable category, and suspension is a self-equivalence of the homotopy category, with quasi-inverse the loop-space functor.

Proposition 4.3 (Properties of $\pi_n^{\mathbb{A}^1}(-)$ and $\pi_n^{\mathsf{Nis}}(-)$ (Exercise 4.20 and Propositions 4.21 and 4.34 in [AE17]))**.**

(i) The Nisnevich, respectively $\mathbb{A}^1$- homotopy sheaves are invariant under Nisnevich-, respectively $\mathbb{A}^1$-local equivalences, i.e. the latter induce isomorphisms on the homotopy sheaves of groups.

(ii) If $X \to \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X$ is a weak equivalence in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$, then the natural map $\pi_n^{\mathsf{Nis}}(X) \to \pi_n^{\mathbb{A}^1}(X)$ is an isomorphism of Nisnevich sheaves for all $n \in \mathbb{N}$.

(iii) If $F \to X$ is the homotopy fiber of $X \to Y$ in $\mathsf{Spc}_{\mathbb{A}^1,*}$, there is a natural long exact sequence of Nisnevich sheaves:

$$\cdots \longrightarrow \pi_{n+1}^{\mathbb{A}^1}(Y) \longrightarrow \pi_n^{\mathbb{A}^1}(F) \longrightarrow \pi_n^{\mathbb{A}^1}(X) \longrightarrow \pi_n^{\mathbb{A}^1}(Y) \longrightarrow \dots$$

(iv) Let $X \in \mathsf{Sm}_S$ be an $\mathbb{A}^1$-rigid scheme. Then $\pi_0^{\mathbb{A}^1}(X) \cong X$ as Nisnevich sheaves, and $\pi_n^{\mathbb{A}^1}(X) \cong 0$ at any basepoint for all $n \in \mathbb{N}^*$.

Proof. The statement of (i) is clear at the level of homotopy presheaves, because $\mathbb{A}^1$-equivalences (respectively Nisnevich equivalences) become isomorphisms in the corresponding homotopy category and thus induce isomorphisms on the hom-sets. Since sheafification is functorial, it carries isomorphisms to isomorphisms, whence our claim.

Item (ii) is a direct consequence of the computation in 4.2 (replacing hom-sets $[-,-]_{\mathbb{A}^1,*}$ by hom-sets $[-,-]_{\mathsf{Nis},*}$ where needed, using the fact that $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X$ is in particular Nisnevich fibrant for any simplicial presheaf X).

To prove (iii), assume that we have a homotopy fiber sequence $F \to X \to Y$ as in the statement. Up to weak equivalence in $\mathsf{Spc}_{\mathbb{A}^1,*}$, we may prove the statement for $X \to Y$ a fibration of fibrant objects, and F the strict pullback of $X \longrightarrow Y \longleftarrow *$ (this is a homotopy pullback by Proposition 3.2). Indeed, by part (i), $\mathbb{A}^1$-equivalences preserve homotopy sheaves. Then, since the identity functor $\mathsf{Spc}_{\mathbb{A}^1,*} \to {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ is right Quillen (the identity functor in the other direction is left Quillen: it preserves cofibrations and weak equivalences by construction), it preserves homotopy limits and $F \to X \to Y$ is also a fiber sequence in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$, satisfying the same hypotheses as the sequence we started from. Given $U \in \mathsf{Sm}_S$, $F(U)$ is therefore the (strict) fiber of the Kan fibration $X(U) \to Y(U)$ (over the basepoint). By Lemma 1.7.3 in [GJ09], we obtain a long exact sequence of groups:

$$\ldots \longrightarrow [S^{n+1}, Y(U)]_{\mathsf{sSet},*} \longrightarrow [S^n, F(U)]_{\mathsf{sSet},*} \longrightarrow [S^n, X(U)]_{\mathsf{sSet},*} \longrightarrow [S^n, Y(U)]_{\mathsf{sSet},*} \longrightarrow \ldots$$

By Remark 4.2, this is precisely the sequence:

$$\ldots \to [\mathcal{S}^{n+1} \wedge U_+, Y]_{\mathsf{Spc}_{\mathbb{A}^1},*} \to [\mathcal{S}^n \wedge U_+, F]_{\mathsf{Spc}_{\mathbb{A}^1},*} \to [\mathcal{S}^n \wedge U_+, X]_{\mathsf{Spc}_{\mathbb{A}^1},*} \to [\mathcal{S}^n \wedge U_+, Y]_{\mathsf{Spc}_{\mathbb{A}^1},*} \to \ldots$$

Taking Nisnevich sheafification at each spot of the sequence yields a long exact exact sequence of homotopy sheaves, because sheafification is exact by Proposition 2.7.

Finally, let us prove item (iv). By Remark 3.10, since X is $\mathbb{A}^1$-rigid, it is already $\mathbb{A}^1$-fibrant. Reproducing the computation in Remark 4.2, we get that $\pi_0^{\mathbb{A}^1}(X)$ is the Nisnevich sheafification of the presheaf defined by $U \in \mathsf{Sm}_S \mapsto [U, X]_{\mathsf{sPre}(\mathsf{Sm}_S)}$. Since both are constant simplicial sets, this is equal to the set of maps of presheaves from U to X, namely $X(U)$. Since X is already a Nisnevich sheaf (because the Nisnevich topology is subcanonical), we get $\pi_0^{\mathbb{A}^1}(X) \cong X$ as desired. For $n \geq 1$, the same computation requires us to consider the presheaf $U \in \mathsf{Sm}_S \mapsto [S^n, X(U)]_{\mathsf{sSet},*}$ which is trivial (no matter the choice of a basepoint) since $X(U)$ is a constant simplicial set. The sheafification of the trivial presheaf being trivial, we are done. □

Note that part (i) does not claim anything about $\mathbb{A}^1$-invariance in the sense of *strict* or *strong* $\mathbb{A}^1$-invariance (see Definition 4.17 below). Morel proved that, when $S = \mathsf{Spec}(k)$ with k a field, $\mathbb{A}^1$-homotopy sheaves are strictly invariant in all dimensions $n \geq 2$ and strongly invariant in dimension 1; he conjectured the case $n = 0$ to be true as well ([Mor12], Theorem 1.9 and Conjecture 1.12). The latter conjecture was only recently disproved ([Ayo23]). This problem at $n = 0$ is only an instance of the difficulties that arise in $\mathbb{A}^1$-homotopy theory about the 0th homotopy sheaves ([WW20], p 26). In general, they can be quite complicated, and it is not obvious how to work "$\mathbb{A}^1$-component by $\mathbb{A}^1$-component", as one would do in topology.

Theorem 4.4 (Unstable $\mathbb{A}^1$-connectivity theorem ([MV99], Section 2, Corollary 3.22 or [AE17], Corollary 4.30 for a proof)). *Assume S is Noetherian of finite Krull dimension. If $X \in {}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$, then the canonical morphism $X \to \mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X$ induces an epimorphism $\pi_0^{\mathbb{N}}(X) \twoheadrightarrow \pi_0^{\mathbb{N}}(\mathbb{L}_{\mathbb{A}^1}\mathbb{L}_{\mathbb{N}}X) \cong \pi_0^{\mathbb{A}^1}(X)$. In particular, the natural map $\pi_0^{\mathsf{Nis}}(\mathsf{Sing}_{\mathbb{A}^1}(X)) \to \pi_0^{\mathbb{A}^1}(X)$ is an epimorphism.*

Proof. Since the natural map $X \to \mathsf{L}_{\mathsf{Nis}}X$ is a Nisnevich local weak equivalence, it induces an isomorphism $\pi_0^{\mathsf{Nis}}(X) \to \pi_0^{\mathsf{Nis}}(\mathsf{L}_{\mathsf{Nis}}X)$ through which factors the map $\pi_0^{\mathsf{Nis}}(X) \to \pi_0^{\mathsf{Nis}}(\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X)$. So it suffice to show the statement for X Nisnevich local.

Since S is Noetherian of finite Krull dimension, by Theorem 3.17, we have $\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}(X) \simeq \mathsf{hocolim}_n\, (\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X)$ in ${}_{\mathsf{S}}\mathsf{Pre}(\mathsf{Sm}_S)$. We can therefore compute:

$$\begin{aligned}\pi_0^{\mathbb{A}^1}(X) &\cong \pi_0(\mathsf{L}_{\mathbb{A}^1}\mathsf{L}_{\mathsf{Nis}}X) \cong \pi_0(\mathsf{hocolim}_n\, (\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X)) \\ &\cong \pi_0(\mathsf{colim}\, (Q_1X \hookrightarrow Q_2X \hookrightarrow \ldots)) \cong \mathsf{colim}_n\, \pi_0(Q_nX) \\ &\cong \mathsf{colim}_n\, \pi_0((\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X)) \cong \mathsf{colim}_n\, \pi_0^{\mathsf{Nis}}((\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X))\end{aligned}$$

where Q_nX is in particular a cofibrant replacement in ${}_{\mathsf{S}}\mathsf{Pre}(Sm_S)$ of $(\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X)$ for $n \geq 1$. So it suffices to show that $\pi_0^{\mathsf{Nis}}(X) \to \pi_0^{\mathsf{Nis}}((\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X))$ is surjective for all $n \geq 1$. Since $(\mathsf{L}_{\mathsf{Nis}} \circ \mathsf{Sing}_{\mathbb{A}^1})^{\circ n}(X)$ is by construction Nisnevich-fibrant for all $n \geq 1$, by induction we only have to show the case $n = 1$ for X Nisnevich-local. By definition of the Nisnevich homotopy presheaves, it suffices to show that $\mathsf{Ho}\,(\mathsf{Spc})\,(U, X) \to \mathsf{Ho}\,(\mathsf{Spc})\,(U, \mathsf{Sing}_{\mathbb{A}^1}X) \cong \mathsf{Ho}\,(\mathsf{Spc})\,(U, \mathsf{L}_{\mathsf{Nis}}\mathsf{Sing}_{\mathbb{A}^1}X)$ is surjective for all $U \in \mathsf{Sm}_S$ (because then Nisnevich sheafification of a surjective map of homotopy presheaves yields a surjective map on the homotopy sheaves). Here is the key computation:

$$\begin{aligned}\mathsf{Ho}\,(\mathsf{Spc})\,(U, \mathsf{Sing}_{\mathbb{A}^1}X) &\cong \mathsf{Ho}\,(\mathsf{Spc})\,(\mathsf{hocolim}_{[m]\in\Delta}\, U \times \Delta^m, X) && (\star) \\ &\cong \pi_0(\mathsf{map}(\mathsf{hocolim}_{[m]\in\Delta}\, U \times \Delta^m, X)) && (\star\star) \\ &\cong \pi_0(\mathsf{holim}_{[m]\in\Delta}\, \mathsf{map}(U \times \Delta^m, X))\end{aligned}$$

where a justification to $(\star\star)$ can be found in the proof of part (iii) of Theorem 4.19, and we justify $(\star)$ just below. This computation allows us to conclude, because $\mathsf{Ho}(\mathsf{Spc})(U, X) \cong \pi_0(\mathsf{map}(U, X))$ surjects onto $\pi_0(\mathsf{holim}_m\ \mathsf{map}(U \times \Delta^m, X))$: indeed, the simplicial set $\mathsf{map}(U, X)$ surjects onto the homotopy limit because it appears as the term for $[m] = [0]$ there. (This is true for strict limits. To generalize it to homotopy limits, realize the homotopy limit as a strict limit; the object indexed by $[0]$ will be replaced by some simplicial set weakly equivalent to $\mathsf{map}(U, X)$, in particular it has the same connected components as $\mathsf{map}(U, X)$).

To justify $(\star)$, we use the left-adjoint $|-|_{\Delta^\bullet \times \underline{\Delta^\bullet}}$ of $\mathsf{Sing}_{\mathbb{A}^1}$ (viewed as a functor $\mathsf{Spc} \to \mathsf{Spc}$) constructed by Morel and Voevodsky (Section 2.3, p 90). It associates to $X \in \mathsf{Spc}$ the coend

$$\int^{\Delta^{op} \times \Delta \ni (n,m)} X_n \times \underline{\Delta^m} \times \Delta^m.$$

We have to show that this adjunction descends to the homotopy categories (so it suffices to show that it is deformable as an adjunction of homotopical categories). On our way we will prove the formula for $|U|_{\Delta^\bullet \times \underline{\Delta^\bullet}}$ used in the "key computation" above.

Choose as right deformation of Spc a fibrant replacement functor, and as left deformation a cofibrant replacement. Since Nisnevich weak equivalences between Nisnevich-local objects are exactly the objectwise equivalences between them, the functor $\mathsf{Sing}_{\mathbb{A}^1}$ preserves them and therefore is compatible with the right deformation we chose. The left adjoint is homotopical on cofibrant objects: for any cofibrant object $X \in \mathsf{Spc}$, we can compute, as in the proof of Lemma 2.32:

$$\begin{aligned}
|X|_{\Delta^\bullet \times \underline{\Delta^\bullet}} &= \int^{\Delta^{op} \times \Delta \ni (n,m)} X_n \times \underline{\Delta^m} \times \Delta^m \\
&\simeq \int^{\Delta^{op} \times \Delta \ni (n,m)} Q_{\mathsf{inj}}(*) \times Q_{\mathsf{proj}}(X_n \times \Delta^m) \\
&\simeq \mathsf{hocolim}_{(n,m) \in \Delta^{op} \times \Delta} X_n \times \Delta^m \qquad \text{(homotopy colimit either in } {}_s\mathsf{Pre}(\mathsf{Sm}_S) \text{ or Spc)} \\
&\simeq \mathsf{hocolim}_{m \in \Delta} (\mathsf{hocolim}_{n \in \Delta^{op}} X_n) \times \Delta^m \qquad \text{(see the end of Remark 2.41)} \\
&\simeq \mathsf{hocolim}_{m \in \Delta} X \times \Delta^m \qquad \text{(for cofibrant X (by the proof of Lemma 2.32))}
\end{aligned}$$

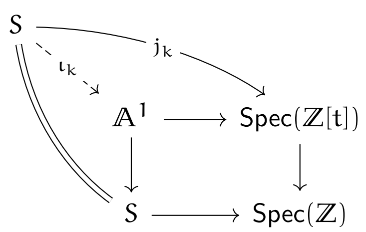
This finishes the proof of $(\star)$.

To show the last part of the statement of the Theorem, recall that $X \to \mathsf{Sing}_{\mathbb{A}^1}$ is an $\mathbb{A}^1$-weak equivalence by Theorem 3.16 and therefore $\pi_0^{\mathbb{A}^1}(X) \cong \pi_0^{\mathbb{A}^1}(\mathsf{Sing}_{\mathbb{A}^1}(X))$. It now suffices to apply the first part of the statement to $\mathsf{Sing}_{\mathbb{A}^1}(X)$.

□

Instead of considering "homotopy classes of maps" as hom-sets in the homotopy category, we can also define actual homotopies between maps and homotopy equivalences.

Definition 4.5 ($\mathbb{A}^1$-homotopic maps)**.** Two maps $f, g : X \to Y$ in $\mathsf{Spc}_{\mathbb{A}^1}$ are called $\mathbb{A}^1$*-homotopic* if there exists a map $H : X \times \mathbb{A}^1 \to Y$ such that $H \circ (\mathrm{id}_X \times \iota_0) = f$ and $H \circ (\mathrm{id}_X \times \iota_1) = g$, where ι_k denotes for $k = 0, 1$ the k-section $S \to \mathbb{A}^1$ obtained via the universal property of the fiber product:



where j_k is induced by the map $\mathbb{Z}[t] \to \Gamma(S, \mathcal{O}_S)$ sending t to k (the zero or the unit element of the ring in question).

A map $f : X \to Y$ is an $\mathbb{A}^1$*-homotopy equivalence* if there exists a map $g : Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the identities.

Lemma 4.6 (Homotopy equivalences are weak equivalences)**.** *Any* $\mathbb{A}^1$*-homotopy equivalence* $f : F \to G$ *is an* $\mathbb{A}^1$*-weak equivalence.*

Proof. Let g be a homotopy inverse for f. To begin with, the k-section $\iota_k : X = X \times S \to X \times \mathbb{A}^1$ is an $\mathbb{A}^1$-weak equivalence for any $X \in \mathsf{Spc}_{\mathbb{A}^1}$ and $k = 0, 1$. This follows from the 2-out-of-3 property since $\pi \circ \iota_k = \mathrm{id}_X$ for $\pi : X \times \mathbb{A}^1 \to X \times S = X$ the projection; and π is an $\mathbb{A}^1$-weak equivalence by Remark 2.41.

By hypothesis, there exist $H : G \times \mathbb{A}^1 \to G$ and $H' : F \times \mathbb{A}^1 \to F$ homotopies between $f \circ g$ and id_G, respectively between $g \circ f$ and id_F. Since $H \circ \iota_1$ and $H' \circ \iota_1$ are the identities by hypothesis, using 2-out-of-3 once more shows that H and H' are $\mathbb{A}^1$-weak equivalences. Then $H \circ \iota_0 = f \circ g$ and $H' \circ \iota_0 = g \circ f$ are $\mathbb{A}^1$-weak equivalences by composition. In particular, for any $\mathbb{A}^1$-local object Z, we have weak equivalences of simplicial sets $\mathsf{map}(f^{op}, Z) \circ \mathsf{map}(g^{op}, Z) : \mathsf{map}(F, Z) \to \mathsf{map}(F, Z)$ and $\mathsf{map}(g^{op}, Z) \circ \mathsf{map}(f^{op}, Z) : \mathsf{map}(F, Z) \to \mathsf{map}(F, Z)$. This implies that $\mathsf{map}(f^{op}, Z)$ and $\mathsf{map}(g^{op}, Z)$ are weak equivalences of simplicial sets as well, and therefore f and g are $\mathbb{A}^1$-weak equivalences by definition. Indeed, looking at the morphisms they induce on the simplicial homotopy groups, the only thing we have to verify is that if $\widetilde{f}$ and $\widetilde{g}$ are morphisms of groups such that $\widetilde{f} \circ \widetilde{g}$ and $\widetilde{g} \circ \widetilde{f}$ are isomorphisms, then both $\widetilde{f}$ and $\widetilde{g}$ are isomorphisms. The first condition implies that $\widetilde{f}$ is surjective and $\widetilde{g}$ is injective, and vice-versa for the second one, whence the claim. □

Example 4.7 (Cylinder object). For any $X \in \mathsf{Spc}_{\mathbb{A}^1}$, $\mathbb{A}^1 \times X$ is a cylinder object for X. Coproducts in $\mathsf{Spc}_{\mathbb{A}^1}$ are computed in the underlying category of simplicial presheaves, i.e. $X \amalg X$ is given by objectwise the disjoint union of simplicial sets. We have a diagram:

$$X \amalg X \xrightarrow{(\mathrm{id}_X \times \iota_0) \amalg (\mathrm{id}_X \times \iota_1)} X \times \mathbb{A}^1 \xrightarrow{\ \pi\ } X, \qquad \nabla : X \amalg X \to X$$

where Δ is the fold map. The projection $X \times \mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-equivalence by Remark 2.41. Commutativity of the diagram is obvious objectwise.

4.2 Classifying spaces

In topology, given a group G, one defines EG as a contractible space with a free action of G. One specific model for EG is given by the CW-complex whose cellular chain complex is the bar resolution of G. The quotient EG/G is called BG, the *classifying space* of G. It is unique up to homotopy and it turns out to be a $K(G, 1)$, namely it is connected and all its homotopy groups are trivial except for the first one, which is isomorphic to G. Following [AE17] and [MV99], we reproduce these constructions in the motivic context (although they are valid for the category of simplicial presheaves on any site ([MV99], section 4.1).

Definition 4.8 (Classifying space)**.** Let $\mathcal{G}$ be a presheaf of groups on Sm_S. Let $E\mathcal{G} \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ be the simplicial presheaf associating to $U \in \mathsf{Sm}_S$ the simplicial bar resolution of the group $\mathcal{G}(U)$, pointed at the neutral element. More explicitly, $E\mathcal{G}(U)_n = \mathcal{G}(U)^{n+1}$ with degeneracies induced by repetition of elements and faces by forgetting elements. We also define a *classifying presheaf* $B\mathcal{G}$ as the levelwise quotient of $E\mathcal{G}$ by the (left) diagonal action of $\mathcal{G}$, with the induced basepoint.

If $\mathcal{G}$ is a simplicial sheaf of groups instead, one defines $E\mathcal{G}$ as the simplicial presheaf given in level n by $E(\mathcal{G}_n)_n$ (using the definition just above), and then the levelwise quotient $B\mathcal{G}$ is given by the diagonal of the bisimplicial object associating to $U \in \mathsf{Sm}_S$ the diagonal of the bisimplicial set $n \mapsto N(\mathcal{G}_n(U))$, where N denotes the nerve of the category associated with the group $\mathcal{G}_n(U)$. Explicitely, this diagonal is given by:

$$* \leftleftarrows \mathcal{G}_1(U) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; \mathcal{G}_2(U) \times \mathcal{G}_2(U) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longleftarrow}}}\; \mathcal{G}_3(U) \times \mathcal{G}_3(U) \times \mathcal{G}_3(U) \quad \cdots$$

with faces given by multiplication of two consecutive elements and applying faces in $\mathcal{G}(U)$.

Viewing a presheaf of groups as a constant simplicial presheaf, this construction agrees with Definition 4.8 (using the "bar notation" for $B\mathcal{G}$).

Restricting ourselves to presheaves of groups viewed as constant simplicial objects instead of considering arbitrary simplicial presheaves of groups is the equivalent of considering only discrete groups in topology. Although it may seem like it is a considerable reduction, the theory is still interesting. Moreover, the application we have in mind for the classifying spaces is in the construction of Postnikov towers in Section 5, where we will take $\mathcal{G} = \pi_1^{\mathbb{A}^1}(X)$ for some pointed simplicial presheaf X, in particular this is only a discrete simplicial sheaf of groups.

If $\mathcal{G}$ is a Nisnevich sheaf of groups, so are $E\mathcal{G}$ and $B\mathcal{G}$: indeed, the equalizers in the definition of a sheaf are computed levelwise in simplicial sets, and a fixed simplicial level of $E\mathcal{G}(U)$ or $B\mathcal{G}(U)$ is just defined by products of $\mathcal{G}(U)$ for all $U \in \mathsf{Sm}_S$. Since limits commute with limits, these products commute with the equalizers, which implies our claim.

These constructions have (some of) the desired properties:

Proposition 4.9 ($B\mathcal{G}$ behaves like a classifying space)**.** *Let $\mathcal{G}$ be a presheaf of groups on Sm_S.*

(i) $E\mathcal{G} \xrightarrow{\sim} *$ *is a weak equivalence in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)_*$ (and thus in $\mathsf{Spc}_{\mathbb{A}^1,*}$ as well).*

(ii) If $\mathcal{G}$ is a sheaf, it holds that $\pi_0^{\mathbb{N}}(B\mathcal{G}) \cong \pi_0^{\mathbb{A}^1}(B\mathcal{G}) \cong *$ (for the statement about the $\mathbb{A}^1$-homotopy sheaf, we assume that S is Noetherian of finite Krull dimension), and $\pi_1^{\mathsf{Nis}}(B\mathcal{G}) \cong \mathcal{G}$, and finally $\pi_n^{\mathsf{Nis}}(B\mathcal{G}) \cong *$ for all $n > 1$. The corresponding statements holds for $\pi_n^{\mathbb{A}^1}(B\mathcal{G})$ when $\mathcal{G}$ is $\mathbb{A}^1$-invariant as a sheaf of groups. More precisely, $B\mathcal{G}$ is Nisnevich-local, and even $\mathbb{A}^1$-local if $\mathcal{G}$ is $\mathbb{A}^1$-invariant.

(iii) If $\mathcal{G}$ satisfies affine Nisnevich excision and (the sheafification) of $\mathcal{G}$ is $\mathbb{A}^1$-invariant on affine schemes, then $\mathcal{G} \to E\mathcal{G} \to B\mathcal{G}$ is a fiber sequence in $\mathsf{Spc}_{\mathbb{A}^1,*}$.

Proof. The result of (i) is standard in the case of groups instead of presheaves of groups; in particular we have an objectwise weak equivalence and thus an equivalence of presheaves. We can define explicitly a simplicial homotopy $E\mathcal{G}(U) \times \Delta^1 \to E\mathcal{G}(U)$ between the identity and the constant map for any $U \in \mathsf{Sm}_S$. To do so, we have to define maps $h_i : E\mathcal{G}(U)_n \to E\mathcal{G}(U)_{n+1}$ for all $0 \leq i \leq n$ and $n \geq 0$ satisfying certain conditions with respect to the faces and degeneracies (as defined in 8.3.11 in [Wei94]). It is straightforward (but tedious) to check that the maps $h_i : \mathcal{G}(U)^n \to \mathcal{G}(U)^{n+1}$, sending a n-uple $(a_1, \ldots, a_n)$ to $(1, 1, \ldots, 1, a_i, \ldots, a_n)$ (with the neutral element $1 \in \mathcal{G}(U)$ in the first i slots), satisfy the conditions of this definition. Moreover, this construction being natural in $U \in \mathsf{Sm}_S$, a "contracting homotopy" $E\mathcal{G} \times \underline{\Delta^1} \to E\mathcal{G}$ exists directly at the level of the presheaves.

In item (ii), the statement for $\mathbb{A}^1$-path components follows from the Nisnevich case and the unstable $\mathbb{A}^1$-connectivity theorem for S Noetherian of finite Krull dimension (Proposition 4.4). For all $U \in \mathsf{Sm}_S$, it is clear from the construction that $\pi_0(B\mathcal{G}(U)) = *$. Therefore, the homotopy groups do not depend on the choice of a basepoint, and $\pi_n(B\mathcal{G}(U))$ is given by $*$ if $n \neq 1$ and $\mathcal{G}(U)$ if $n = 1$, by the properties of the classifying space construction for simplicial sets (see for instance [GJ09], Sections V.4 and V.7). So we are done if we show that $B\mathcal{G}$ is Nisnevich local, respectively $\mathbb{A}^1$-local when $\mathcal{G}$ is $\mathbb{A}^1$-invariant. Indeed, in this case, the computation of Remark 4.2 shows that the Nisnevich-, respectively $\mathbb{A}^1$- homotopy presheaves are computed objectwise. The classifying space is objectwise fibrant; this is again a property of the simplicial construction (also see [GJ09]). Also, this construction is at the level of simplicial sets a right Quillen functor from the category of simplicial groups to ${}_S\mathsf{Set}$, it also preserves weak equivalences (Theorem 7.8 in [GJ09]). In particular, we can check the Nisnevich descent condition: if $U_\bullet = \coprod_{i \in I_\bullet} \mathsf{Sm}_S(-, R_{\bullet,i}) \to V$ is an hypercover, then

$$B\mathcal{G}(V) \to \mathsf{holim}_{n \in \mathbb{N}} \prod_{i \in I_n} B\mathcal{G}(R_{n,i}) \simeq B\left(\mathsf{holim}_{n \in \mathbb{N}} \prod_{i \in I_n} \mathcal{G}(R_{n,i})\right)$$

is a weak equivalence because $\mathcal{G}(V) \simeq \mathsf{holim}_{n \in \mathbb{N}} \prod_{i \in I_n} \mathcal{G}(R_{n,i})$ by Proposition A.6 (since $\mathcal{G}$ is a Nisnevich sheaf by assumption).

If $\mathcal{G}$ is $\mathbb{A}^1$-invariant then the projection induces a group isomorphism $\mathcal{G}(U) \to \mathcal{G}(\mathbb{A}^1 \times_S U)$ for all $U \in \mathsf{Sm}_S$, by definition. The classifying space construction from the category of groups to ${}_S\mathsf{Set}$ preserves isomorphisms because it is functorial. The $\mathbb{A}^1$-invariance of $B\mathcal{G}$ follows.

We will not prove item (iii), but we explain how to deduce it from a result in [AHW18]. In this article, Theorem 2.2.5 states that a fiber sequence in ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)_*$ remains a fiber sequence in the $\mathbb{A}^1$-local Jardine model structure (see Section 6.1 and more precisely Theorem 6.3) in particular if the last term has affine Nisnevich excision and its 0th simplicial homotopy sheaf is $\mathbb{A}^1$-invariant on affine schemes. Here the last term is $B\mathcal{G}$; which has affine Nisnevich excision if $\mathcal{G}$ does, by similar arguments as in part (ii). The 0th homotopy sheaf of $B\mathcal{G}$ is trivial also by part (ii). Finally, note that by Theorem 6.3, a fiber sequence in the $\mathbb{A}^1$-local Jardine model structure is also a fiber sequence in $\mathsf{Spc}_{\mathbb{A}^1,*}$ (because the identity functor in this direction is right Quillen).

□

Classifying spaces will appear again in Section 5. As another illustration of their importance, we cite a result of [MV99]. We first need a definition:

Definition 4.10 ($\mathcal{G}$-torsor)**.** Let $\mathcal{G}$ be a (simplicial) Nisnevich sheaf of groups. A morphism of simplicial presheaves $E \to X$, together with a (levelwise) free action of $\mathcal{G}$ on E over X, is called a *$\mathcal{G}$-torsor* if the levelwise quotient defines an isomorphism of simplicial presheaves $E/\mathcal{G} \to X$.

From the definition it is clear that the quotient $E\mathcal{G} \to B\mathcal{G}$ is a $\mathcal{G}$-torsor. It is universal in the sense described below:

Proposition 4.11 (Universal $\mathcal{G}$-torsor (Section 4, Lemma 1.12 in [MV99]))**.** *Let $\mathcal{G}$ be a Nisnevich sheaf of groups and $\mathcal{E} \to X$ be a $\mathcal{G}$-torsor in ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)$. Then there is an objectwise acyclic fibration $\mathcal{Y} \xrightarrow{\sim} X$ in ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)$, which is also a stalkwise acyclic fibration, and a map $Y \to B\mathcal{G}$ such that the pullback of $\mathcal{E} \to X$ along $\mathcal{Y} \to X$ is isomorphic to the pullback of $E\mathcal{G}$ to $\mathcal{Y}$:*

$$\begin{array}{ccccc} E\mathcal{G} & \longleftarrow & E\mathcal{G} \times_{B\mathcal{G}} \mathcal{Y} \cong \mathcal{E} \times_X \mathcal{Y} & \longrightarrow & \mathcal{E} \\ \downarrow & \llcorner & \downarrow & \lrcorner & \downarrow \\ B\mathcal{G} & \underset{\exists}{\longleftarrow} & \mathcal{Y} & \underset{\exists}{\overset{\sim}{\twoheadrightarrow}} & X \end{array}$$

Up to replacing the base with a weakly equivalent one, any $\mathcal{G}$-torsor is therefore a pullback of the "universal $\mathcal{G}$-torsor" $E\mathcal{G} \to B\mathcal{G}$.

Proof. Let $\mathcal{Y}$ be the objectwise quotient of $\mathcal{E} \times E\mathcal{G}$ by the diagonal action of $\mathcal{G}$. Since the action of $\mathcal{G}$ on both $\mathcal{E}$ and $E\mathcal{G}$ is free by construction, as a $\mathcal{G}$-module, $(\mathcal{E} \times E\mathcal{G})^{\Delta}$ (Δ denotes the diagonal action) is isomorphic to the same underlying presheaf with action either on the first factor or second factor. In particular, we have isomorphisms of presheaves

$$\mathcal{Y} = (\mathcal{E} \times E\mathcal{G})^{\Delta} \Big/ \mathcal{G} \cong \mathcal{E}/\mathcal{G} \times E\mathcal{G} \cong \mathcal{E} \times E\mathcal{G}/\mathcal{G}.$$

and the two last presheaves are respectively isomorphic to $X \times E\mathcal{G}$ and $\mathcal{E} \times B\mathcal{G}$ by definition. In particular, as a map $\mathcal{Y} \to X$ we choose the projection $X \times E\mathcal{G} \to X$, which is clearly a trivial fibration, $E\mathcal{G}$ being objectwise fibrant and contractible. For the isomorphism of the pullbacks, we compute:

$$E\mathcal{G} \times_{B\mathcal{G}} \mathcal{Y} \cong E\mathcal{G} \times_{B\mathcal{G}} (B\mathcal{G} \times \mathcal{E}) \cong E\mathcal{G} \times \mathcal{E} \cong \mathcal{E} \times_X (X \times E\mathcal{G}) \cong \mathcal{E} \times_X \mathcal{Y}$$

An objectwise fibration is in particular a stalkwise fibration ([Jar87a], end of p 38). Moreover, it is a standard fact that a fibration of simplicial sets is acyclic if and only if its fiber over any vertex is contractible. Thus we only have to check that, stalkwise, all fibers of $\mathcal{Y} \to X$ are contractible. Objectwise, the fibers are all given by the corresponding sections of $E\mathcal{G}$. We have defined in the proof of Proposition 4.9 an objectwise simplicial null-homotopy of $E\mathcal{G}$, which is also natural. We want to argue that this null-homotopy passes to the stalks. Since the stalk functor commutes with finite limits by definition, it preserves fibers. Also, taking a product with $\Delta^1 \in {}_{S}\mathsf{Set}$ commutes with the colimit that defines the stalk (apply the fact that the stalk functor preserves products to $E\mathcal{G} \times \underline{\Delta^1}$). In particular, these natural null-homotopies pass to the colimit and define a null-homotopy of any stalk of the fiber. This concludes the proof. □

4.3 Eilenberg-MacLane spaces

Given an abelian group G, there exists a topological space $K(G,n)$, called an *Eilenberg-MacLane space*, unique up to homotopy, that has a single non trivial homotopy group, isomorphic to G, in dimension n. For the case $n = 0$, we may just choose the group itself, viewed as a discrete topological space. Uniqueness up to homotopy is proved using the Hurewicz theorem and the fact that Eilenberg-MacLane spaces represent singular cohomology of spaces (the homotopy classes of maps from a CW-complex X to $K(G,n)$ are in bijection with $H^n(X;G)$). The spaces BG give an example of $K(G,1)$ spaces (even when G is not abelian). We shall now construct Eilenberg-MacLane spaces in the motivic set-up. The separated treatment of the objects $B\mathcal{G}$ in subsection 4.2 is still relevant, because we did not restrict ourselves to presheaves of *abelian* groups. As in subsection 4.2, the constructions have nothing specific to the motivic setting, and can be performed with respect in the category of simplicial presheaves on any site. As usual everything will be based on simplicial constructions. The following theorem will be useful to us:

Theorem 4.12 (Dold-Kan correspondence, Dold-Puppe theorem ([Kan58], also see Section 22 in [May92], and [Qui67], Section II.4, item (5) on page 4.11))**.** *Let* Ab *be the category of abelian groups, and* $\mathsf{Ab}_\bullet$ *be the category of non-negatively graded complexes of abelian groups. There is an equivalence of categories:*

$$N : {}_S\mathsf{Ab} \rightleftarrows \mathsf{Ab}_\bullet : \Gamma$$

which in particular is an adjunction $N \dashv \Gamma$*. This adjunction is a Quillen pair with respect to the Quillen model structure on simplicial groups and the projective model structure on chain complexes. The functor* Γ *preserves all weak equivalences. And:*

(i) the functor N *sends* $\mathcal{A}_\bullet \in {}_S\mathsf{Ab}$ *to the complex* $\{\mathcal{A}_n \cap \ker(\partial_0) \cap \cdots \cap \ker(\partial_{n-1})\}_{n\geq 0}$*, with differentials* $d_n = (-1)^n \partial_n$ *for all* $n \geq 1$*, where* ∂_i *are the faces of* $\mathcal{A}_\bullet$ *as a simplicial object.*

(ii) The functor Γ *sends a complex* A_* *to the simplicial object given in degree* n *by*

$$\bigoplus_{\substack{[n]\to[k] \\ \text{surjective} \\ \text{monotone}}} V_k.$$

A monotone map $f : [m] \to [n]$ *in* Δ *is sent to the map* $\Gamma(A_*)_m \to \Gamma(A_*)_n$*, given on the summand corresponding to some monotone surjection* $g : [n] \to [k]$ *by* $A_k \to A_s$*, where* $s = |g([m]) + 1|$ *and* A_s *is the summand corresponding to the monotone surjection* $[m] \to [s]$ *that factors* $f \circ g$ *as a monotone surjection followed by a monotone injection.*

This theorem still holds if we replace Ab by any other abelian category, except that the explicit description of the function Γ we give has to be adapted. We are interested in applying this theorem to the category of presheaves of abelian groups on Sm_S. We may do so because (pre)sheaves on any site, with values in an abelian category, form themselves an abelian category (see the Stacks project, Tag 03CM). The description of Γ in this case is given levelwise by the corresponding construction for abelian groups.

Definition 4.13 (Eilenberg-MacLane object)**.** Let $\mathcal{G}$ be a presheaf of abelian groups on Sm_S. For any $n \in \mathbb{N}$, the *Eilenberg-MacLane object* $K(\mathcal{G},n)$ is the simplicial presheaf of abelian groups on Sm_S sending $U \in \mathsf{Sm}_S$ to $\Gamma(\mathcal{G}(U)[n])$, where $\mathcal{G}(U)[n]$ denotes the complex of simplicial abelian groups with only $\mathcal{G}(U)$ in degree n. It sends a morphism of schemes $f : U \to V$ to the map induced in each simplicial level and on each summand by $\mathcal{G}(f)$. We choose the neutral element of this simplicial group as a basepoint.

Since the complex $\mathcal{G}(U)[n]$ is concentrated in degree n, the description of Theorem 4.12 simplifies:

- For all $k < n$, we have $K(\mathcal{G},n)_k = \{0\}$, since there is no surjection $[k] \to [n]$.
- At simplicial level n, we have $K(\mathcal{G},n)_n = \mathcal{G}$, because the only monotone surjection $[n] \to [n]$ is the identity.
- For all $k > n$, $K(\mathcal{G},n)_k = \bigoplus_{0<j_1<\cdots<j_n\leq k} \mathcal{G}$. Indeed, there are exactly $\binom{k}{n}$ monotone surjections g from $[k] \to [n]$, corresponding to the choice of the subset $\{x \in \{1,\ldots,k\} \mid g(x) = g(x-1)+1\}$ (illustration below).

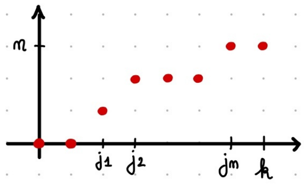


Figure 5: An illustration with $n = 3$ and $k = 7$.

The degeneracy $d_\ell : K(\mathcal{G}, n)_k \to K(\mathcal{G}, n)_{k+1}$ sends the summand corresponding to a sequence $(j_1, \dots, j_n)$ identically to the summand indexed by the sequence $(j_1, \dots, j_{\ell'-1}, j_{\ell'} + 1, \dots, j_n + 1)$ where ℓ' is the smallest index such that $j_{\ell'} > \ell$. Indeed, this degeneracy corresponds in Δ to the map $[k+1] \to [k]$ repeating ℓ, it is already surjective, so the injection in the factorization is the identity and we will get the identity on $\mathcal{G}$, and the composition $[k+1] \to [k] \to [n]$ is the surjection represented by the sequence $(j_1, \dots, j_{\ell'-1}, j_{\ell'} + 1, \dots, j_n + 1)$.

The face $\partial_\ell : K(\mathcal{G}, n)_{k+1} \to K(\mathcal{G}, n)_k$ is zero on summands corresponding to sequences $(j_1, \dots, j_n)$ such that both ℓ and $\ell + 1$ appear in the sequence $(0, j_1, \dots, j_n, k+2)$: indeed, then the composition $[k] \to [k+1] \to [n]$ that corresponds is not surjective, so $s < n$ in Theorem 4.12 above, and $\mathcal{G}[n]_s = 0$. Otherwise, the face sends this summand identically to the summand indexed by the sequence $(j_1, \dots, j_\ell, j_{\ell+1} - 1, \dots, j_n - 1)$ (the composition $[k] \to [k+1] \to [n]$ is now surjective and corresponds to this sequence).

Example 4.14 ($K(\mathcal{G}, 1)$ is a model for $B\mathcal{G}$). With the explicit description given above, it is not too hard to see that if $\mathcal{G}$ is a sheaf of abelian groups, then $B\mathcal{G} \cong K(\mathcal{G}, 1)$. Indeed, the simplices are $\{0\}$ at level 0, and $K(\mathcal{G}, 1)_n$ is a direct sum of copies of $\mathcal{G}$ indexed by sequences with a single element in $\{1, \dots, k\}$, so we recognize $B\mathcal{G}$ for $n \geq 1$. If we denote by $\mathcal{G}_k$ the copy indexed by the integer k, then the degeneracies and faces are:

$$d_\ell : K(\mathcal{G}, 1)_n \longrightarrow K(\mathcal{G}, 1)_{n+1}$$
$$\mathcal{G}_k \longmapsto \begin{cases} \mathcal{G}_k & \text{if } k \leq \ell \\ \mathcal{G}_{k+1} & \text{if } k > \ell \end{cases}$$

$$\partial_\ell : K(\mathcal{G}, 1)_{n+1} \longrightarrow K(\mathcal{G}, 1)_n$$
$$\mathcal{G}_k \longmapsto \begin{cases} 0 & \text{if } k = 1,\ \ell = 0 \\ \mathcal{G}_k & \text{if } k \leq \ell \leq n \\ \mathcal{G}_{k-1} & \text{if } \ell + 1 \leq k \leq n \\ 0 & \text{if } k = n + 1 = \ell \end{cases}$$

which corresponds to the faces given by multiplication (addition in the abelian case) for $B\mathcal{G}$, because both $\mathcal{G}_\ell$ and $\mathcal{G}_{\ell+1}$ are mapped to $\mathcal{G}_\ell$ in the target (for $\ell \leq n$).

This is the motivic analog to the fact that the classifying space of a discrete topological group G is a $K(G, 1)$ (we are considering our presheaves of groups as constant simplicial objects, and constant simplicial sets are the analogs of discrete topological spaces).

Lemma 4.15 (Simplicial homotopy sheaves of $K(\mathcal{G}, n)$)**.** *Let* $n \in \mathbb{N}$ *and* $\mathcal{G}$ *be a* sheaf *of abelian groups on* Sm_S*. Then* $K(\mathcal{G}, n)$ *is a sheaf of Kan complexes with simplicial homotopy presheaves* $\pi_k(K(\mathcal{G}, n)(-)) = 0$ *for all* $k \neq n$*, and* $\pi_n(K(\mathcal{G}, n)(-)) \cong \mathcal{G}(-)$*.*

Proof. The functor Γ in Theorem 4.12 is right Quillen, so it preserves fibrant objects. Since every object is fibrant in the projective model structure on chain complexes, Γ takes its values in Kan complexes. So $K(\mathcal{G}, n)$ is objectwise a Kan complex. Moreover, Γ preserves limits, in particular the equalizer and products appearing in the definition of a sheaf. Since $\mathcal{G}$ is a sheaf, this implies that $K(\mathcal{G}, n)$ is a sheaf as well.

The k-th simplicial homotopy presheaf is given by $U \in \mathsf{Sm}_s \mapsto \pi_k(\Gamma(\mathcal{G}(U)[n])$. What we will prove is that if $\mathcal{A}$ is a fibrant simplicial abelian group, then $\pi_n(\mathcal{A}, 0) \cong H_n(N\mathcal{A})$, using the functor N of Theorem 4.12 ([GJ09], Section III, p 153 before Corollary 2.7). If this holds, for $n > 0$, note that $\Gamma(\mathcal{G}(U)[n])$ has a single 0-simplex and therefore there is only one possible basepoint to compute the homotopy groups. And for all $k \in \mathbb{N}$, $\pi_k(\Gamma(\mathcal{G}(U)[n])) \cong H_k(N\Gamma(\mathcal{G}(U)[n])) \cong H_k(\mathcal{G}(U)[n])$ because $N \circ \Gamma$ is naturally isomorphic to the identity by Theorem 4.12, and this homolopy group is $\mathcal{G}(U)$ if $k = n$ and $\{0\}$ else, as desired (these are already the simplicial homotopy sheaves since $\mathcal{G}$ is a sheaf by assumption). The case $n = 0$ can be treated without the claim: then $K(\mathcal{G}, 0)$ is just the discrete simplicial set $\mathcal{G}$.

To prove the claim, recall that the homotopy group $\pi_k(A, 0)$ is given by the homotopy classes k-simplices of A whose boundary lies in the degeneracies of the vertex 0. The quotient by the equivalence relation of being homotopic defines a group homomorphism. Thus the homotopy group is the quotient of the subgroup of k-simplices whose boundary is trivial by the subgroup of nullhomotopic k-simplices. The set of k-simplices of $\mathcal{A}$ whose boundary is the vertex 0 is by definition $\mathcal{A}_k \cap \mathsf{ker}(\partial_0) \cap \cdots \cap \mathsf{ker}(\partial_k)$, which is the kernel of the differential $N(\mathcal{A})_k \to N(\mathcal{A})_{k-1}$ by construction. The image of the differential $N(\mathcal{A})_{k+1} \to N(\mathcal{A})_k$ consists of the simplices $y \in \mathcal{A}_k$ whose boundary is the vertex 0 and such that y is $(k+1)$-th face of a $k+1$-simplex whose other faces are all trivial. These correspond exactly to k-simplices in $\mathcal{A}$ that are nullhomotopic for the equivalence relation described above. This identifies the quotient groups $\pi_k(\mathcal{A})$ and $H_n(N\mathcal{A})$. □

Remark 4.16. We have just seen that the Eilenberg-MacLane objects we constructed have the correct simplicial homotopy sheaves and presheaves. However, we will see in Theorem 4.19) that they do not have the correct Nisnevich homotopy *presheaves*, although the problem disappears for homotopy sheaves. This comes from the fact that $K(\mathcal{G}, n)$ is a priori not Nisnevich local. One might hope for a similar argument as in the proof of Theorem 5.7 (paragraph "Homotopy groups"), using the fact that Γ preserves homotopy limits. However, these are homotopy limits in the category of chain complexes, and for $n \in \mathbb{N}$, the functor $\mathcal{A} \in \mathsf{Ab} \mapsto \mathcal{A}[n] \in \mathsf{Ab}_\bullet$ does not preserve homotopy limits...

Before stating our result for Nisnevich homotopy sheaves, it will turn out useful to study the relation between Eilenberg-MacLane spaces and sheaf cohomology. We first define stronger notions of $\mathbb{A}^1$-invariance (recall Definition 2.43) for sheaves of groups, by viewing them as coefficients:

Definition 4.17 (Strictly and strongly $\mathbb{A}^1$-invariant)**.** Let $\mathcal{G}$ be a Nisnevich sheaf of groups on Sm_S.

- It is called *strongly $\mathbb{A}^1$-invariant* if for any $U \in \mathsf{Sm}_S$, the projection induces isomorphisms on the 0th and 1st cohomology groups with $\mathcal{G}$ coefficients: $H^i_{\mathsf{Nis}}(U, \mathcal{G}) \cong H^i_{\mathsf{Nis}}(U \times_S \mathbb{A}^1, \mathcal{G})$, $i \in \{0, 1\}$.
- It is called *strictly $\mathbb{A}^1$-invariant* if it is a sheaf of *abelian* groups and the projection induces isomorphisms as in the previous bullet point for *all* $i \in \mathbb{N}$.

Cohomology is defined here in the usual sense of sheaf cohomology: $H^i_{\mathsf{Nis}}(U, -)$ is the i-th right derived functor of the global sections functor. The latter associates to a Nisnevich sheaf of groups $\mathcal{G}_U$ on U (with the Zariski topology) the group $\mathcal{G}_U(U)$ (a Nisnevich sheaf $\mathcal{G}$ on Sm_S can be viewed as a Nisnevich sheaf $\mathcal{G}_U$ over some smooth scheme $U \in \mathsf{Sm}_S$ by restriction to Sm_U).

As in topology, Eilenberg-MacLane objects are closely related with cohomology:

Proposition 4.18 ($K(\mathcal{G}, n)$ represents cohomology ([Bro73], Section 3, Theorem 2))**.** *Let $\mathcal{G}$ be a sheaf of abelian groups on Sm_S, where S is Noetherian of finite Krull dimension. Then for any $U \in \mathsf{Sm}_S$, there is a canonical isomorphism:*

$$\mathsf{Ho}\,(\mathsf{Spc})\,(U, K(\mathcal{G}, n)) \cong H^n(U, \mathcal{G})$$

where $H^n(U, \mathcal{G})$ is the usual sheaf cohomology (in the Nisnevich topology) defined as the n-th right derived functor of the global section functor (on U).

Proof. **Step 1.** We first show for all $U \in \mathsf{Sm}_S$ that the cohomology $H^n(U, \mathcal{G})$ can be computed as $\mathsf{Ho}\,(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)\,((\mathbb{Z}_U[0])^+, \mathcal{G}[n])$, where:

- $\mathsf{AbShv}(\mathsf{Sm}_S)$ is the category of sheaves of abelian groups on Sm_S, and $\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet$ is the corresponding category of chain complexes, with weak equivalences the maps inducing isomorphisms in homology (quasi-isomorphisms). A subtlety here is that this is defined in the category of *sheaves*, namely the homology is not defined objectwise; it has to be sheafified.
- $\mathcal{G}[n]$ is the complex with only $\mathcal{G}$ in degree n.
- $\mathbb{Z}_U$ is the presheaf of abelian groups such that $\mathsf{AbPre}(\mathsf{Sm}_S)(\mathbb{Z}_U, \mathcal{F}) \cong \mathcal{F}(U)$ for all $\mathcal{F} \in \mathsf{AbPre}(\mathsf{Sm}_S)$. More precisely, given $V \in \mathsf{Sm}_S$, $\mathbb{Z}_U(V)$ is the free abelian group on the set $U(V) = \mathsf{Sm}_S(V, U)$.

- $(-)^+$ denotes (degreewise) sheafification.

This is actually the definition of hypercohomology (see Appendix A in [Voe03]), and it coincides with the usual definition of sheaf cohomology as we show now. Firstly, morphisms $(\mathbb{Z}_U[0])^+ \to \mathcal{G}[n]$ in $\mathsf{Ho}(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)$ are represented by homotopy classes of maps $(\mathbb{Z}_U[0])^+ \to \mathcal{I}^{n-\bullet}$ in $\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet$ where $\mathcal{G} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{G}$. Indeed, in the injective model structure on chain complexes, every object is cofibrant and an injective resolution yields a fibrant replacement for $\mathcal{G}$.

A morphism of chain complexes $(\mathbb{Z}_U[0])^+ \to \mathcal{I}^{n-\bullet}$ is the data of a morphism of sheaves of abelian groups $(\mathbb{Z}_U)^+ \to \mathcal{I}^n$ whose post-composition by the differential of $\mathcal{I}^{n-\bullet}$ is 0, or equivalently a morphism of presheaves of abelian groups $\mathbb{Z}_U \to \mathcal{I}^n$ whose post-composition by the differential is 0. By construction of $\mathbb{Z}_U$ this corresponds exactly to an element of $\mathcal{I}^n(U)$ whose image in $\mathcal{I}^{n+1}(U)$ by the differential is trivial. Such a map is nullhomotopic exactly when it factors through $\mathcal{I}^{n-1}(U)$, namely when the corresponding section lies in the image of $\mathcal{I}^{n-1}(U)$ by the differential. From this description, we get $\mathsf{Ho}(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)((\mathbb{Z}_U[0])^+, \mathcal{G}[n]) \cong H_n(\mathcal{I}^{n-\bullet}(U)) \cong H^n(\mathcal{I}^\bullet(U))$ but the latter is by definition the sheaf cohomology $H^n(U, \mathcal{G})$.

Step 2. We now use the adjunction of Theorem 4.12 to make the Eilenberg-MacLane object appear. Note that $\mathbb{Z}_U[0] \cong N(\mathbb{Z}_U)$ where $\mathbb{Z}_U$ is viewed as a constant simplicial object. Therefore, the computation we would like to perform is the following one:

$$\begin{aligned} H^n(U, \mathcal{G} &\cong \mathsf{Ho}(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)([N(\mathbb{Z}_U)]^+, \mathcal{G}[n]) \\ &\cong \mathsf{Ho}({}_s\mathsf{AbPre}(\mathsf{Sm}_S))(\mathbb{Z}_U, K(\mathcal{G}, n)) && (\star) \\ &\cong \mathsf{Ho}({}_s\mathsf{Pre}(\mathsf{Sm}_S)_{\mathcal{J}})(U, K(\mathcal{G}, n)) && (\star\star) \\ &\cong \mathsf{Ho}(\mathsf{Spc})(U, K(\mathcal{G}, n)) \end{aligned}$$

The steps $(\star)$ (a generalized Dold-Kan correspondence) and $(\star\star)$ (a generalization of the free-forgetful adjunction between sets and abelian groups) have to be justified; this is done in Lemmata 4.22 and 4.23 below. Here, ${}_s\mathsf{Pre}(\mathsf{Sm}_S)_{\mathcal{J}}$ denotes the fact that we work with stalkwise weak equivalences (like the Jardine model structure from Definition 6.1). The last bijection in the computation above then follows from the Quillen equivalence with Spc when S is Noetherian of finite Krull dimension, see Theorem 6.2. □

We can use similar methods to compute the homotopy sheaves of Eilenberg-MacLane spaces:

Theorem 4.19 (Nisnevich- and $\mathbb{A}^1$- homotopy sheaves of $K(\mathcal{G}, n)$)**.** *Let $n \in \mathbb{N}$ and $\mathcal{G}$ be a* sheaf *of abelian groups on Sm_S, where S is Noetherian of finite Krull dimension. Then:*

(i) For all $U \in \mathsf{Sm}_S$ and $k \geq 0$, we have $\pi_k(L_{\mathsf{Nis}}K(\mathcal{G}, n)(U), 0) = H^{n-k}_{\mathsf{Nis}}(U, \mathcal{G})$, where cohomology groups in negative dimensions are defined to be zero.

(ii) For S Noetherian of finite Krull dimension, $\pi^{\mathsf{Nis}}_k(K(\mathcal{G}, n)) = 0$ for all $k \neq n$ and $\pi^{\mathsf{Nis}}_n(K(\mathcal{G}, n)) = \mathcal{G}$.

(iii) The sheaf $\mathcal{G}$ is strictly $\mathbb{A}^1$-invariant if and only if $L_{\mathsf{Nis}}K(\mathcal{G}, n)$ is $\mathbb{A}^1$-invariant for all $n \in \mathbb{N}$.

(iv) In the situation of item (iii), part (ii) also holds for $\mathbb{A}^1$-homotopy sheaves.

Remark 4.20. One might worry about $\mathbb{A}^1$-invariance of $L_{\mathsf{Nis}}K(\mathcal{G}, n)$ not being well defined. Indeed, our construction of L_{Nis} (2.28) was not explicit and a priori allows for other possibilities. Assume L'_{Nis} is another functor built in the same manner. Then for any $X \in {}_s\mathsf{Pre}(\mathsf{Sm}_S)$, $L_{\mathsf{Nis}}X$ is $\mathbb{A}^1$-invariant if and only if $L'_{\mathsf{Nis}}X$ is $\mathbb{A}^1$-invariant. Indeed, we have a zigzag $L'_{\mathsf{Nis}}X \leftarrow X \to L_{\mathsf{Nis}}X$ of Nisnevich weak equivalences, so $\mathsf{Ho}(\mathsf{Spc})(L'_{\mathsf{Nis}}X, L_{\mathsf{Nis}}X)$ contains an isomorphism. By construction, both objects are fibrant and cofibrant in Spc and therefore the maps between them in the homotopy category all admit a lift $L'_{\mathsf{Nis}}X \to L_{\mathsf{Nis}}X$ in Spc, which is a Nisnevich equivalence if and only if it becomes an isomorphism in the homotopy category. Thus, we have two Nisnevich-equivalent Nisnevich-local objects: they must be objectwise equivalent. In particular, $\mathbb{A}^1$-invariance of one of them implies $\mathbb{A}^1$-invariance of the other.

Proof of Theorem 4.19.

Part (i). For all $U \in \mathsf{Sm}_S$ and $k \geq 0$, we can compute:

$$\begin{aligned} \pi_k(L_{\mathsf{Nis}}K(\mathcal{G}, n)(U), 0) &\cong [\mathcal{S}^k \wedge U_+, L_{\mathsf{Nis}}K(\mathcal{G}, n)]_{{}_s\mathsf{Pre}(\mathsf{Sm}_S),*} \cong [\mathcal{S}^k \wedge U_+, L_{\mathsf{Nis}}K(\mathcal{G}, n)]_{\mathsf{Spc},*} \\ &\cong [\mathcal{S}^k \wedge U_+, K(\mathcal{G}, n)]_{\mathsf{Spc},*} = \mathsf{Ho}(\mathsf{Spc}_*)(\mathcal{S}^k \wedge U_+, K(\mathcal{G}, n)) \end{aligned}$$

as in Remark 4.2. We will use a method similar to the one in Step 2 of the proof of 4.18. By Lemma 4.23 and Lemma 4.22, we have

$$\begin{aligned}\mathsf{Ho}\,(\mathsf{Spc}_*)\,(\mathcal{S}^k \wedge U_+, K(\mathcal{G}, n)) &\cong \mathsf{Ho}\,({}_{S}\mathsf{AbPre}(\mathsf{Sm}_S))\,(\mathbb{Z}_{\mathcal{S}^k \wedge U_+}/\mathbb{Z}, K(\mathcal{G}, n)) \\ &\cong \mathsf{Ho}\,(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)\,([N(\mathbb{Z}_{\mathcal{S}^k \wedge U_+}/\mathbb{Z})]^+, \mathcal{G}[n])\end{aligned}$$

We have also used the Quillen equivalence between Spc and ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)_{*,\mathcal{J}}$ of Theorem 6.2 when S is Noetherian of finite Krull dimension. We will show that $\mathbb{Z}_{\mathcal{S}^k \wedge U_+}/\mathbb{Z} \cong \Gamma(\mathbb{Z}_U[k])$, and then the above is isomorphic to $\mathsf{Ho}\,(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)\,((\mathbb{Z}_U[k])^+, \mathcal{G}[n]) \cong \mathsf{Ho}\,(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)\,((\mathbb{Z}_U[0])^+, \mathcal{G}[n-k])$ because the functors N and Γ are quasi-inverses equivalences of categories (Theorem 4.12). Finally, the above is isomorphic to $H^{n-k}(U, \mathcal{G})$ by Step 1 in the proof of Proposition 4.15.

So to conclude we only have to compute that $\mathbb{Z}_{\mathcal{S}^k \wedge U_+}/\mathbb{Z} \cong \Gamma(\mathbb{Z}_U[k])$, where the inclusion of $\mathbb{Z}$ in $\mathbb{Z}_{\mathcal{S}^k \wedge U_+}$ is induced by the inclusion of the basepoint. We only write the computation for the simplices (and one can check that the identifications we will do are compatible with the faces and degeneracies). We recall that our model for $S^k \in {}_{S}\mathsf{Set}$ is the quotient $\Delta^k/\partial\Delta^k$.

- In level $m < k$, we have $S^k_m = \{*\}$ and therefore $(\mathcal{S}^k \wedge U_+)_m = *$ the (degeneracy of the) basepoint. In particular, $(\mathbb{Z}_{\mathcal{S}^k \wedge U_+})_m/\mathbb{Z}_m = \mathbb{Z}/\mathbb{Z} \cong \{0\}$. On the other hand, $\Gamma(\mathbb{Z}_U[k])_m = \{0\}$ since there are no surjections $[m] \twoheadrightarrow [k]$ (see the description in Theorem 4.12).
- In level $m \geq k$, $S^k_m = \{f : [m] \twoheadrightarrow [k] \mid f \text{ surjective monotone}\} \sqcup \{*\}$ because $\Delta^k_m = \Delta([m], [k])$ and all such maps except for the surjective ones can be written as degeneracies of non-identity maps in $\Delta([k], [k])$, namely simplices of $\partial\Delta^k$. The added basepoint $\{*\}$ represents all the simplices that have been collapsed (there is at least on them). Therefore, $(\mathcal{S}^k \wedge U_+(-))_m = (\{f : [m] \twoheadrightarrow [k]\} \times U(-))_+$ and finally

$$(\mathbb{Z}_{(\mathcal{S}^k \wedge U_+)})_m/(\mathbb{Z}_m) = \left(\left(\oplus_{\{f:[m]\twoheadrightarrow[k]\}} \mathbb{Z}_U\right) \oplus \mathbb{Z}_*\right)\Big/\mathbb{Z}_* \cong \bigoplus_{\{f:[m]\twoheadrightarrow[k]\}} \mathbb{Z}_U = \Gamma(\mathbb{Z}_U[k])_m.$$

Part (ii). Using part (i) and the definition of Nisnevich homotopy sheaves, it suffice to show that the presheaf $U \in \mathsf{Sm}_S \mapsto H^k(U, \mathcal{G})$ sheafifies to zero unless $k = 0$. When $k = 0$, this is by definition the presheaf $U \in \mathsf{Sm}_S \mapsto \mathcal{G}(U)$, but since $\mathcal{G}$ is already a sheaf, the sheafification is just $\mathcal{G}$. Let $k \neq 0$. A presheaf has trivial sheafification if and only if all its stalks are zero (in a site with enough points, which is the case for Sm_S when S is Noetherian of finite Krull dimension; see subsection A.4). Let p be a point of the site Sm_S and $\bullet_p : \mathcal{F} \mapsto \mathsf{colim}_{(U,x)}\, \mathcal{F}(U)$ be the associated stalk functor. Therefore, we can compute, for $\mathcal{I}^\bullet$ an injective resolution of $\mathcal{G}$:

$$H^k(-, \mathcal{G})_p = H^k(\mathcal{I}^\bullet(-))_p = H^k(\mathcal{I}^\bullet(-)_p) = 0$$

because the stalk functor associated to a point of a site is always exact (Lemma 18.36.1, part (3) in the Stacks project, Tag 04EM), so it commutes with homology (Exercise 1.6.H. in [Vak17], the FHHF theorem). Moreover $\mathcal{I}^\bullet$ is exact except in dimension 0 since it is a resolution.

Part (iii). We take our inspiration from [AM11] (below Definition 4.3.1) and [Bac24] (Lemma 1.5), although the statement already appears as Remark 1.8 in [Mor12].

By Proposition 4.18, $H^n_{\mathsf{Nis}}(U, \mathcal{G}) \cong \mathsf{Ho}\,(\mathsf{Spc})\,(U, K(\mathcal{G}, n)) \cong \mathsf{Ho}\,(\mathsf{Spc})\,(U, L_{\mathsf{Nis}}K(\mathcal{G}, n))$ for all $n \in \mathbb{N}$. By definition, $L_{\mathsf{Nis}}K(\mathcal{G}, n)$ is $\mathbb{A}^1$-invariant if and only for all $U \in \mathsf{Sm}_S$, the projection induces a weak equivalence $\mathsf{map}(U, L_{\mathsf{Nis}}K(\mathcal{G}, n)) \to \mathsf{map}(U \times_S \mathbb{A}^1, L_{\mathsf{Nis}}K(\mathcal{G}, n))$.

We first observe that $\pi_0(\mathsf{map}(X, Y)) \cong \mathsf{Ho}\,(\mathcal{C})\,(X, Y)$ for X cofibrant and Y fibrant, where $\mathcal{C}$ is either ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)$, Spc or $\mathsf{Spc}_{\mathbb{A}^1}$. Indeed, in this situation $\mathsf{Ho}\,(\mathcal{C})\,(X, Y)$ can be described by the homotopy classes of maps in $\mathcal{C}(X, Y) = \mathsf{map}(X, Y)_0$ with respect to a fixed good cylinder object for X. For all three categories, we can choose $X \amalg X \hookrightarrow X \times \underline{\Delta^1} \to X$. This is because the projection $X \times \underline{\Delta^1} \to X$ is an objectwise weak equivalence and $X \amalg X = X \times \underline{\Delta^0} \hookrightarrow X \times \underline{\Delta^1}$ is a cofibration by the pushout product axiom applied to the cofibration $\emptyset \hookrightarrow X$ and $\underline{\Delta^0} \hookrightarrow \underline{\Delta^1}$ (end of Remark 2.36). But the equivalence relation this defines exactly corresponds to the equivalence relation $\sim$ on $\mathsf{map}(X, Y)_0$ such that $\pi_0(\mathsf{map}(X, Y)) = \mathsf{map}(X, Y)_0/\sim$.

If $L_{\mathsf{Nis}}K(\mathcal{G}, n)$ is $\mathbb{A}^1$-invariant for all $n \in \mathbb{N}$, the map induced on the connected components by $\mathsf{map}(U, L_{\mathsf{Nis}}K(\mathcal{G}, n)) \to \mathsf{map}(U \times_S \mathbb{A}^1, L_{\mathsf{Nis}}K(\mathcal{G}, n))$ is a bijection. Since U is cofibrant and $L_{\mathsf{Nis}}K(\mathcal{G}, n)$

is fibrant in Spc, this exactly means that $\mathrm{Ho}(\mathsf{Spc})(U, \mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n)) \to \mathrm{Ho}(\mathsf{Spc})(U \times_S \mathbb{A}^1, \mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n))$ is a bijection, so $\mathcal{G}$ is strictly $\mathbb{A}^1$-invariant by our previous observations.

Conversely, we want weak equivalences $\mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n)(U) \to \mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n)(U \times_S \mathbb{A}^1)$ for all $n \in \mathbb{N}$ and $U \in \mathsf{Sm}_S$. The projection induces isomorphisms on the homotopy groups at the basepoint 0. Indeed, this follows directly from part (i) and the strict $\mathbb{A}^1$-invariance of $\mathcal{G}$. Then, the projection induces isomorphisms on homotopy groups at all basepoints, because $\mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n)(U)$ and $\mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n)(U \times_S \mathbb{A}^1)$ are groups up to homotopy (more precisely, groups objects in the homotopy category ([Bac24], proof of Lemma 1.5)). Then the multiplication law allows to perform translations that are weak equivalences, and thus isomorphisms on homotopy at a fixed basepoint are enough to characterize weak equivalences. This additional structure of group objects comes from the fact that $K(\mathcal{G},n)$ is a sheaf of simplicial abelian groups, and $\mathsf{L}_{\mathsf{Nis}}$ preserves products up to (objectwise) weak equivalence (proof of Corollary 4.28 in [AE17]), and $\mathsf{L}_{\mathsf{Nis}}(*) \simeq *$. Therefore $\mathsf{L}_{\mathsf{Nis}}$ preserves products and the terminal object at the level of the homotopy category, in particular it preserves group objects there. Since weak equivalences are defined objectwise in ${}_{S}\mathsf{Pre}(\mathsf{Sm}_S)$, we get this structure of group up to homotopy also objectwise.

Part (iv). In this situation, $\mathsf{L}_{\mathsf{Nis}}K(\mathcal{G},n)$ is Nisnevich fibrant and $\mathbb{A}^1$-invariant, i.e. it is $\mathbb{A}^1$-local. The claim follows since Nisnevich- and $\mathbb{A}^1$- homotopy sheaves coincide for $\mathbb{A}^1$-local spaces.

□

Remark 4.21 (Strong $\mathbb{A}^1$-invariance and $B\mathcal{G}$). Similarly to part (iii) in Theorem 4.19, a sheaf of (non-necessarily abelian) groups $\mathcal{G}$ is strongly $\mathbb{A}^1$-invariant if and only if $\mathsf{L}_{\mathsf{Nis}}B\mathcal{G}$ is $\mathbb{A}^1$-invariant (same references as for strict invariance).

We used the following Lemmata in the computations above. Our references for these statements are [Mor12] (Section 6.2, p 161), [Jar87b] (Chapitre 8, p 192 and Lemma 8.4) and [Bro73] (Section 3, observations (1) to (5) before Theorem 2).

Lemma 4.22 (Generalized Dold-Kan correspondence)**.** *Assume S is Noetherian of finite Krull dimension. Then the Dold-Kan adjunction of Theorem 4.12 for the abelian category* $\mathsf{AbShv}(\mathsf{Sm}_S)$ *descends to the homotopy categories as an adjunction:*

$$N : \mathrm{Ho}({}_{S}\mathsf{AbShv}(\mathsf{Sm}_S)) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Ho}(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet) : \Gamma$$

for the weak equivalences given respectively by stalkwise weak equivalences and quasi-isomorphisms (with sheafified homology). In particular, we have for all $X \in {}_{S}\mathsf{AbPre}(\mathsf{Sm}_S)$ *and* $Y \in \mathsf{AbShv}(\mathsf{Sm}_S)_\bullet$ *a bijection:*

$$\mathrm{Ho}({}_{S}\mathsf{AbPre}(\mathsf{Sm}_S))(X, \Gamma(Y)) \cong \mathrm{Ho}(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet)((NX)^+, Y).$$

Proof. For the first part of the statement, it suffices to show that the functors N and Γ between the category of simplicial sheaves of abelian groups and complexes of abelian sheaves both preserve weak equivalences (see for instance [Bro73], p 426, the Adjoint Functor Lemma).

For N, note that the corresponding functor at the level of abelian simplicial groups sends weak equivalences to quasi-isomorphisms, and commutes with colimits as a left adjoint. This means that for $\mathcal{A}$ a simplicial abelian sheaf, and p a point in Sm_S with associated stalk functor $\bullet_p$, we have $(N\mathcal{A})_p \cong N(\mathcal{A}_p) \in {}_{S}\mathsf{Set}$. In particular, our functor N for simplicial abelian sheaves sends objectwise weak equivalences to stalkwise quasi-isomorphisms. These are the quasi-isomorphisms in the category of complexes of abelian sheaves, because the stalk functor is exact and therefore commutes with homology, and an isomorphism at all stalks is an isomorphism of sheaves when the site has enough points (which holds for Sm_S when S is Noetherian of finite Krull dimension (subsection A.4)). This argument also shows that N reflects weak equivalences: if $\mathcal{A} \to \mathcal{B}$ is a morphism of simplicial abelian sheaves, and $N\mathcal{A} \to N\mathcal{B}$ is a quasi-isomorphism of complexes of abelian sheaves, then we have that for any point p of the site, $N(\mathcal{A}_p) \cong (N\mathcal{A})_p \to (N\mathcal{B})_p \cong N(\mathcal{B}_p)$ is a quasi-isomorphism of chain complexes of abelian groups. Since Γ preserves weak equivalences of complexes of abelian groups, and $\Gamma \circ N$ is naturally isomorphic to the identity, we obtain a weak equivalence of simplicial abelian groups $\mathcal{A}_p \to \mathcal{B}_p$. So $\mathcal{A} \to \mathcal{B}$ is stalkwise a weak equivalence. But then if $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ is a weak equivalence of complexes of abelian sheaves, we have that $\Gamma(\mathcal{A}_\bullet) \to \Gamma(\mathcal{B}_\bullet)$ is a weak equivalence if and only if $N\Gamma(\mathcal{A}_\bullet) \to N\Gamma(\mathcal{B}_\bullet)$ is a weak equivalence. Using that $N \circ \Gamma$ is naturally isomorphic to the identity, we obtain that Γ preserves (and reflects) weak equivalences as well.

For the second part of the statement, we compute for $X \in {}_S\mathsf{AbPre}(\mathsf{Sm}_S)$ and $Y \in \mathsf{AbPre}_\bullet(\mathsf{Sm}_S)$:

$$\begin{aligned}\mathsf{Ho}\left({}_S\mathsf{AbPre}(\mathsf{Sm}_S)\right)(X, \Gamma(Y)) &\cong \mathsf{Ho}\left({}_S\mathsf{AbShv}(\mathsf{Sm}_S)\right)(X^+, \Gamma(Y)) \\ &\cong \mathsf{Ho}\left(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet\right)(N(X^+), Y) \\ &\cong \mathsf{Ho}\left(\mathsf{AbShv}(\mathsf{Sm}_S)_\bullet\right)((NX)^+, Y) \qquad \text{(since } N \text{ preserves stalks)}\end{aligned}$$

The first isomorphism follows from the fact that the sheafification-forgetful adjunction descends to the homotopy categories. Once more, this is because sheafification clearly preserves stalkwise weak equivalences (since it preserves stalks), and the forgetful functor from sheaves to presheaves preserves them too. □

Lemma 4.23 (Free-forgetful adjunctions for Spc or Spc_* and ${}_S\mathsf{AbPre}(\mathsf{Sm}_S)$)**.** *The free-forgetful adjunction between ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ and ${}_S\mathsf{AbPre}(\mathsf{Sm}_S)$ induces adjunctions on the homotopy categories:*

$$\mathbb{Z}_{(-)} : \mathsf{Ho}\left({}_S\mathsf{Pre}(\mathsf{Sm}_S)_{\mathcal{J}}\right) \rightleftarrows \mathsf{Ho}\left({}_S\mathsf{AbPre}(\mathsf{Sm}_S)\right) : U \quad (\perp)$$
$$X \longmapsto \mathbb{Z}_X$$

$$\mathbb{Z}_{(-)}/\mathbb{Z}_* : \mathsf{Ho}\left({}_S\mathsf{Pre}(\mathsf{Sm}_S)_{*,\mathcal{J}}\right) \rightleftarrows \mathsf{Ho}\left({}_S\mathsf{AbPre}(\mathsf{Sm}_S)\right) : U_0 \quad (\perp)$$
$$(X, x_0) \longmapsto \mathbb{Z}_X/\mathbb{Z}_{x_0}$$
$$(U(\mathcal{A}), 0) \longleftarrow\!\shortmid \mathcal{A}$$

($\mathbb{Z}_X$ is defined as in the proof of Proposition 4.18) where the weak equivalences are the stalkwise weak equivalences (this is what we denote by $\mathcal{J}$, like the Jardine model structure from Definition 6.1).

Proof. For the non-pointed case, note that before localization the functor U clearly preserves stalkwise weak equivalences. The functor $\mathbb{Z}_{(-)}$ from simplicial sets to simplicial abelian groups is left adjoint to the forgetful functor, therefore it commutes with colimits. This means that for X a simplicial presheaf, and p a point in Sm_S with associated stalk functor $\bullet_p$, we have $(\mathbb{Z}_X)_p \cong \mathbb{Z}_{(X_p)} \in {}_S\mathsf{Set}$. Therefore the induced functor on simplicial presheaves preserves stalkwise weak equivalences if and only if $\mathbb{Z}_{(-)}$ preserves weak equivalences of simplicial sets. The latter property holds because we can consider simplicial sets with their usual Quillen model structure, and simplicial abelian groups with the induced weak equivalences and fibrations (but cofibrations defined by the lifting property). Since every simplicial set is cofibrant, it suffices by K. Brown's lemma to check that the free construction preserves acyclic cofibrations. This holds because it is left adjoint to U which preserves fibrations and acyclic fibrations. As in the proof of Lemma 4.22 this implies that the adjunction descends to the homotopy categories (without deriving the functors).

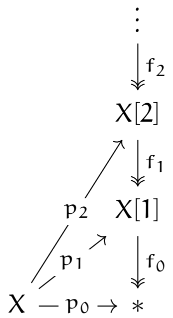
For the pointed case, we first note that these functors define an adjunction before localization. Indeed, by construction, a morphism of simplicial abelian presheaves $\mathbb{Z}_X/\mathbb{Z}_{x_0} \to \mathcal{A}$ for (X, x_0) some pointed simplicial presheaf is exactly a morphism $\mathbb{Z}_X \to \mathcal{A}$ whose pre-composition with the inclusion $\mathbb{Z}_{x_0} \to \mathbb{Z}_X$ is trivial. By adjunction in the non-pointed case, this corresponds exactly to a morphism of simplicial presheaves $X \to U(\mathcal{A})$ sending x_0 to $0 \in \mathcal{A}$, namely a pointed morphism of simplicial presheaves. To verify that the adjunction passes to the homotopy categories, since U_0 clearly preserves stalkwise weak equivalences, we just have to check as above that the functor $\mathbb{Z}_{(-)}/\mathbb{Z}_*$ from pointed simplicial sets to simplicial abelian groups preserves weak equivalences. This can be proved in the exact same way as in the non-pointed case we explained above. □

5 Postnikov towers

5.1 In topology

For this subsection we follow [Hat02], Section 4.3. Given a topological space, we may want to approximate it with spaces that do not have too many non trivial homotopy groups, in a similar fashion that one approximates a CW-complex by considering its n-skeleton to do arguments by induction on the dimension of the cells, or study its low-dimensional (co)homology. This can be done as follows:

Theorem 5.1 (Existence of Postnikov towers)**.** *Let* X *be a pointed path-connected CW-complex. Then,* X *admits a* Postnikov tower, *i.e. there exists a tower of pointed path-connected spaces:*



satisfying the following properties:

(i) For all $i \in \mathbb{N}$*, the map* $p_i : X \to X[i]$ *induces isomorphisms on all homotopy groups in dimension* $k \leq i$*, and* $\pi_k(X[i]) = 0$ *for all* $k > i$*.*

(ii) All maps f_i *for* $i \in \mathbb{N}$*, are fibrations (with respect to the Quillen model structure on* Top*), whose homotopy fiber is a* $K(\pi_{i+1}(X), i+1)$*.*

(iii) The natural map $X \to \lim_i X[i] \simeq \operatorname{holim}_i X[i]$ *induced by all* $\{p_i\}$ *is a weak homotopy equivalence.*

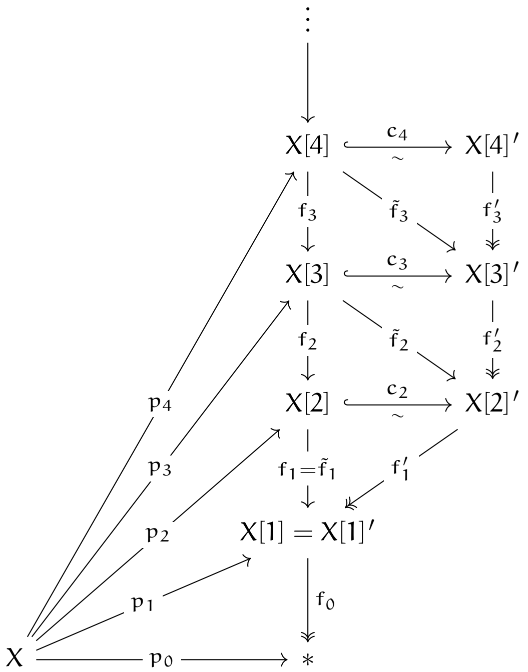
One reason to ask for path-connectedness is that a pointed map of pointed path-connected spaces is a weak equivalence if and only if induces isomorphisms on the homotopy groups *at the basepoints* in all dimensions. If X is not path-connected, one may apply the above theorem to each path-connected component of X, but then one has to choose a basepoint in each component. Moreover, if we are willing to work up to weak equivalence, we may drop the assumption that X is a CW-complex, and apply the theorem to a CW-replacement for X. Indeed, there always exists a CW-complex Y and with a weak homotopy equivalence $Y \to X$ (but because this approximates X from the *left*, a Postnikov tower for Y does not directly induce a Postnikov tower for X).

Proof. **Step 1:** construction of the spaces $X[i]$ and the maps $\{p_i\}$. We set $X[0] = *$, since X is path-connected, we have a bijection on the set of connected components. Let $i \in \mathbb{N}^*$. We will attach cells to X to kill all homotopy groups in dimensions $k \geq i+1$. Define inductively on $k \geq i$ a space $X[i]^{(k)}$ where $X[i]^{(i)} = X$ and $X[i]^{(k+1)}$ is obtained from $X[i]^{(k)}$ as follows: pick a set of generators for $\pi_{k+1}(X[i]^{(k)})$, and choose maps $\mathbb{S}^{k+1} \to X[i]^{(k)}$ representing these generators. Then, $X[i]^{(k+1)}$ is the CW-complex obtained by attaching a $(k+2)$-cell along each map $\mathbb{S}^{k+1} \to X[i]^{(k)}$ we picked. By construction, the $(k+1)$-th homotopy group of $X[i]^{(k+1)}$ is trivial, and in all lower dimensions the inclusion $X[i]^{(k)} \to X[i]^{(k+1)}$ induces an isomorphism on the homotopy groups. Set $X[i] = \operatorname{colim}_{k \geq i} X[i]^{(k)} = \bigcup_{k \geq i} X[i]^{(k)}$. Then, let p_i be the natural map $X = X[i]^{(i)} \to X[i]$. For all $j \in \mathbb{N}^*$, since $\mathbb{S}^j$ is compact and $X[i]$ is obtained by a sequential colimit of closed inclusions, we have $\pi_j(X[i]) = \pi_j(\operatorname{colim}_{k \geq i} X[i]^{(k)}) = \operatorname{colim}_{k \geq i} \pi_j(X[i]^{(k)})$, which is isomorphic to $\pi_j(X)$ if $j \leq i$ and 0 else, by construction (we may always ignore a finite number of terms in the colimit). Moreover, the map that p_i induces on the homotopy groups is exactly the natural map $\pi_j(X[i]^{(i)}) = \pi_j(X) \to \operatorname{colim}_{k \geq i} \pi_j(X[i]^{(k)})$, so it is an isomorphism in dimensions $j \leq i$, as desired, and the homotopy groups of $X[i]$ are trivial in dimensions greater than i.

Step 2: construction of the maps $\{f_i\}$ and replacement by fibrations. Let $i \geq 1$. We have CW-inclusions $p_i : X \to X[i]$ and $p_{i+1} : X \to X[i+1]$. We aim at defining $f_i : X[i+1] \to X[i]$ such that

$f_i \circ p_{i+1} = p_i$. We will do so by induction on the skeleton of $X[i+1]$. The $(i+2)$-th skeleton of $X[i+1]$ being that of X by construction, we just define the restriction of f_i to this skeleton to be p_i. If the map has been defined on the k-skeleton for some integer $k \geq i+2$, consider a $(k+1)$-cell in $X[i+1]$ and its attaching map $\mathbb{S}^k \to X[i+1]$. Then, the post-composition by f_i on the k-skeleton $\mathbb{S}^k \to X[i+1]_k \to X[i]$ is null-homotopic because $\pi_k(X[i]) = 0$ since $k > i$. A null-homotopy being exactly the data of a map from the cone on $\mathbb{S}^k$ to $X[i]$ that extends this composition, viewing the cell attached as forming a cone on $\mathbb{S}^k$, we can extend f_i over this cell. By induction, this defines f_i on all cells of $X[i+1]$.

To replace the maps $\{f_i\}$ by fibrations, we again proceed by induction on i. The map $f_0 : X[1] \to *$ is a fibration since all topological spaces are fibrant. We replace the next stages by using acyclic cofibration – fibration factorizations in the model category Top as follows:



The maps are build inductively starting from the bottom of the tower. Let $c_1 = \mathrm{id}_{X[1]}$. At each stage, $f'_i \circ c_{i+1}$ is a factorization of $\tilde{f}_i$ as an acyclic cofibration followed by a fibration, and then we set $\tilde{f}_{i+1} = c_{i+1} \circ f_{i+1}$. We take $p'_i = c_i \circ p_i$. Since c_i is a weak equivalence, the desired properties of the homotopy groups that we had before are still verified.

Step 3: homotopy fibers. Since $f_i : X[i+1] \to X[i]$ is a fibration for all $i \geq 1$, letting E be its (homotopy) fiber, we have a long exact sequence:

$$\cdots \to \pi_{j+1}(X[i+1]) \to \pi_{j+1}(X[i]) \to \pi_j(E) \to \pi_j(X[i+1]) \to \pi_j(X[i]) \to \ldots$$

Setting $j = i+1$, the sequence becomes $0 \to \pi_{i+1}(E) \to \pi_{i+1}(X[i+1]) \to 0 = \pi_{i+1}(X[i])$, so $\pi_{i+1}(E) \cong \pi_{i+1}(X)$. If $j \neq i, i+1$, then $\pi_j(X[i+1]) \to \pi_j(X[i])$ and $\pi_{j+1}(X[i+1]) \to \pi_{j+1}(X[i])$ are isomorphisms, thus $\pi_j(E) = 0$. For $j = i$, $\pi_j(X[i+1]) \to \pi_j(X[i])$ is still an isomorphism and $\pi_{j+1}(X[i]) = 0$ thus $\pi_i(E) = 0$. Namely, E is a $K(\pi_{i+1}(X), i+1)$.

Step 4: convergence. We will use Proposition 4.67 in [Hat02]: For any sequence of fibrations $\cdots \to X_2 \to X_1$ and $i \in \mathbb{N}$, the natural map $\pi_i(\lim_n X_n) \to \lim_n \pi_i(X_n)$ is a surjection. Moreover, if there exists $M \in \mathbb{N}$ such that the maps $\pi_{i+1}(X_{n+1}) \to \pi_{i+1}(X_n)$ are surjective for all $n \geq M$, then the natural map of the previous sentence is an isomorphism.

The proposition is not too hard to show and uses crucially the homotopy lifting property of the fibrations appearing in the limit. Here, the maps $\pi_{i+1}(X[n+1]) \to \pi_{i+1}(X[n])$ are isomorphisms for $n \geq i+1$ so the homotopy groups commute with the limit. By construction, the natural map from $\pi_i(X)$ to the limit of the groups $\{\pi_i(X[n])\}_{n \geq 1}$ is clearly an isomorphism, whence the conclusion.

□

One of the interests of Postnikov towers is that in good situations they can be used to construct maps with target X (or one of the stages $X[n]$) by inductively lifting maps along the tower. This is possible in particular when the tower consists of *principal fibrations*. In this situation, one obtain a nice *obstruction theory*, describing the existence of lifts in terms of the vanishing of certain characteristic cohomology classes.

Definition 5.2 (Principal fibration)**.** A fibration $E \twoheadrightarrow B$ in Top with homotopy fiber F is called principal if there is a commutative diagram:

$$\begin{array}{ccccccc} & & F & \longrightarrow & E & \longrightarrow & B \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \Omega B' & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

where the second row is a fibration sequence (each map is the homotopy fiber of the one following it) and the vertical maps are weak equivalences as indicated on the diagram.

Note that not every fibration is principal in Top. When working in the category of spectra, one could pick $B' = \Sigma F$ and we would get a diagram as in the definition, using that cofiber sequences are the same as fiber sequences, and that the compositions $\Sigma\Omega$ and $\Omega\Sigma$ are stably equivalent to the identity. This is not true anymore in Top.

Theorem 5.3 (Existence of a Postnikov tower of principal fibrations)**.** *The tower of Theorem 5.1 for some path-connected CW-complex X can be chosen to be a tower of principal fibrations if and only if $\pi_1(X)$ acts trivially on the higher homotopy groups of X. In particular, if X is simply connected, then X admits a Postnikov tower of principal fibrations.*

Proof. We show that the condition is necessary to give some intuition, and a proof that it is sufficient can be found in [Hat02], Theorem 4.69. Recall that the action of the fundamental group on higher homotopy groups is given as follows: pick pointed maps $f : \mathbb{S}^k \to X$ and $\gamma : \mathbb{S}^1 \to X$ representing elements of $\pi_k(X, x_0)$ and $\pi_1(X, x_0)$ respectively, where x_0 is a chosen basepoint for X. We shall define a map $\gamma \cdot f : \mathbb{S}^k \to X$ whose homotopy class will represent $[\gamma] \cdot [f] \in \pi_k(X, x_0)$. Since $(\mathbb{S}^k, *)$ is a CW-pair, the map $* \to \mathbb{S}^k$ is a cofibration and thus we have a lift with respect to the path-space fibration (this is the homotopy extension property):

$$\begin{array}{ccc} * & \xrightarrow{\ \gamma\ } & X^I \\ \downarrow & \overset{\ell}{\nearrow} & \wr\big\downarrow \mathrm{ev}_0 \\ \mathbb{S}^k & \xrightarrow{\ f\ } & X \end{array}$$

Then, we define $[\gamma] \cdot [f] \in \pi_k(X, x_0)$ as the the homotopy class of the composition of ℓ and evaluation at 1 (the map $\mathrm{ev}_1 \circ \ell$ is homotopic to f, but the action is non trivial because this homotopy doesn't fix basepoints. Actually, the basepoint follows during the homotopy the path γ).

If X admits a tower of principal fibrations, there is a fiber sequence $X[i+1] \to X[i] \to B_i$ for some space B_i, for all $i \in \mathbb{N}$. Then, $\pi_1(X[i+1])$ acts trivially on $\pi_n(X[i], X[i+1])$ (basepoints are implicit). Indeed, looking at the long exact sequences for a fiber sequence and for relative homotopy groups, $\pi_n(X[i], X[i+1]) \cong \pi_n(B_i)$, and the action of $\pi_1(X[i+1])$ is trivial since $\pi_1(X[i+1]) \to \pi_1(B_i)$ is trivial (it is the composition of two successive maps in the long exact sequence). But we have $\pi_1(X[i+1]) = \pi_1(X)$, and $\pi_{i+2}(X[i], X[i+1]) = \pi_{i+1}(X)$ because of the long exact sequence:

$$\cdots \to \underbrace{\pi_{i+2}(X[i])}_{=0} \to \pi_{i+2}(X[i], X[i+1]) \to \underbrace{\pi_{i+1}(X[i+1])}_{=\pi_{i+1}(X)} \to \underbrace{\pi_{i+1}(X[i])}_{=0} \to \dots$$

□

In the case of a principal fibration, B' must be a $K(\pi_{i+1}(X), i+2)$ (in the notation of Definition 5.2). Indeed, $\Omega B'$ must then be a $K(\pi_{i+1}(X), i+1)$ by Theorem 5.1 (it is equivalent to the homotopy fiber of $X[i+1] \to X[i]$). Therefore, $\pi_k(B') \cong \pi_{k-1}(\Omega B')$ is equal to $\pi_{i+1}(X)$ if $k = i+2$ and $\{0\}$ otherwise (and B' is path-connected since its loop space is path-connected). In particular, we have a map $X[i] \to K(\pi_{i+1}(X), i+2)$ determined up to weak equivalence, and actually up to homotopy by the Whitehead theorem if we choose CW-complex models for our Eilenberg-MacLane spaces, and consider the original X[i]'s before replacement (they were CW-complexes as well). Since Eilenberg-MacLane spaces represent singular cohomology for CW-complexes, this is equivalent to the data of a cohomology class $k_i \in H^{i+2}(X[i], \pi_{i+1}(X))$ which is called the i-th k-invariant. These invariants describe the obstructions to lifting a map through the successive stages of the tower, as the following result shows:

Theorem 5.4 (Obstruction theory in Postnikov towers)**.** *Let X be a path-connected CW-complex with a Postnikov tower $\{X[i]\}_{i\geq 1}$ with principal fibrations. Then, given a CW-complex Y and $i \geq 1$, a map $f : Y \to X[i]$ lifts to a map $Y \to X[i+1]$ if and only if $f^*(k_i) = 0 \in H^{i+2}(Y, \pi_{i+1}(X))$.*

Proof. By hypothesis, we have the following situation:

$$\begin{array}{ccccc} & & K(\pi_{i+1}(X), i+1) \longrightarrow & X[i+1] & \longrightarrow * \simeq K(\pi_{i+1}(X), n+2)^I \\ & \overset{g}{\nearrow} & & \downarrow & \downarrow \\ Y & \xrightarrow{\quad f \quad} & & X[i] & \xrightarrow{\; k_i \;} K(\pi_{i+1}(X), n+2) \end{array}$$

where the square on the right is a homotopy pullback (a strict pullback if we consider the path space fibration instead of the inclusion of the point) since $X[i+1] \to X[i] \to K(\pi_{i+1}(X), n+2)$ is a fiber sequence. Note that $f^*(k_i)$ is the cohomology class corresponding by representability of cohomology to the (homotopy class) of the map $k_i \circ f$. Thus it vanishes if and only if $k_i \circ f$ is null-homotopic.

If a lift g exists, then $k_i \circ f$ factors through the path space $K(\pi_{i+1}(X), n+2)^I$ (or, using the point of view of homotopy pullbacks, is homotopic to a map that factors through the point). It is null-homotopic and $f^*(k_i)$ vanishes. Conversely, if $k_i \circ f$ is null-homotopic, this exactly means that it factors through the path space fibration for $K(\pi_{i+1}(X), n+2)$ and therefore the universal property of the pullback provides the desired lift (we could also say that the trivial map $Y \to *$ creates a square that commutes up to homotopy and the universal property of the homotopy pullback gives the desired lift). ☐

Dually, one can also define Whitehead towers, approximating a space X "from the left", by n-connected spaces, namely spaces whose homotopy groups are all trivial until a certain dimension n (whereas Postnikov towers approximate it "from the right" by spaces whose homotopy groups are trivial from a certain dimension on, i.e. n-truncated spaces). Another "dual" construction to Postnikov towers are *homology decompositions*, which as the name indicates approximate the homology of a space instead of its homotopy (see [Hil65], Chapter 8).

5.2 In the simplicial world

Postnikov-type constructions can be described in a wide variety of contexts. The construction we did in Top can be reproduced in the category of spectra (of topological spaces) for instance; more generally one can define a notion of Postnikov systems in any triangulated category (where we think of distinguished triangles as fiber sequences). We can also define Postnikov towers in categories of simplicial presheaves over a (well-behaved) Grothendieck site, and this is the situation that will be of interest to us, in the case of the site $(\mathsf{Sm}_S, \tau_{Nis})$. The way we performed most "topological constructions" in $\mathsf{Spc}_{\mathbb{A}^1}$ so far, is by considering the corresponding construction for simplicial sets, applying it objectwise (and then sometimes taking Nisnevich sheafification). This is the reason why we need to know about Postnikov towers in the category of simplicial sets before passing to the motivic setting. Following [May92] (Section 8) and [MV99] (p 57), here is one construction in the simplicial world:

Definition 5.5 (Simplicial Postnikov tower)**.** Let $X \in {}_S\mathsf{Set}$. For all $i \in \mathbb{N}$, the i*-th Postnikov truncation (or section) of* X is the simplicial set X[i] with n-simplices $X[i]_n = X_n / \sim_{i+1}$ where $\sim_{i+1}$ is the equivalence relation on X_n such that for any two n-simplices x and y, viewed as simplicial maps $\Delta^n \to X$, we have:

$$x \sim_{i+1} y \iff x|_{\mathsf{sk}_{i+1}(\Delta^n)} = y|_{\mathsf{sk}_{i+1}(\Delta^n)}.$$

Then X[i] is viewed as a quotient of X, and its faces and degeneracies are induced by those of X.

Alternatively, X[i] can be defined as the image of the natural map $X \to \mathsf{cosk}_{i+1}(X)$.

If X is pointed, X[i] naturally has an induced basepoint.

The two definitions are equivalent because $x|_{\mathsf{sk}_{i+1}(\Delta^n)}, y|_{\mathsf{sk}_{i+1}(\Delta^n)} : \mathsf{sk}_{i+1}(\Delta^n) \to X$ are equal if and only if their adjoints $x^\sharp, y^\sharp : \Delta^n \to \mathsf{cosk}_{i+1}(X)$ are equal, but the latter are exactly the images of x and y by the natural map $X \to \mathsf{cosk}_{i+1}(X)$.

This is the simplicial construction used all the references that give some details on how to build the Postnikov tower we found. However, when trying to write down precise proofs, it seemed more convenient to use another model. Instead of considering as n-th truncation the image of the natural map $X \to \mathsf{cosk}_{n+1}(X)$, we can directly use $\mathsf{cosk}_{n+1}(X)$ as the n-th stage of the tower. Thus there are more simplices in the second construction. The latter is only better-behaved with respect to some aspects, and loses some good properties, however this will suffice for our purposes:

Lemma 5.6 (Comparison of the properties of $\mathsf{cosk}_{n+1}(X)$ and $X[n]$)**.** *Let $n \in \mathbb{N}$.*

(i) If X is a Kan complex, $X[n]$ and $\mathsf{cosk}_{n+1}X$ are Kan complexes.

(ii) The inclusion $X[n] \to \mathsf{cosk}_{n+1}X$ is a weak equivalence, and the natural map $X \to X[n]$ induces isomorphisms on all homotopy groups in dimensions $k \leq n$; moreover $\pi_k(X[n])$ is trivial at all basepoints for all $k > n$.

(iii) The natural map $X[n+1] \to X[n]$ is a Kan fibration, but the natural map $\mathsf{cosk}_{n+2}X \to \mathsf{cosk}_{n+1}X$ might not be a fibration.

(iv) The functor cosk_{n+1} preserves limits; it might not hold for the functor $X \mapsto X[n]$. Both functors preserve weak equivalences. In particular, cosk_{n+1} preserves homotopy limits.

(v) A map $f : X \to Y$ in $_{\mathsf{S}}\mathsf{Set}$ is a weak equivalence if and only if $f[n] : X[n] \to Y[n]$ is a weak equivalence for all $n \in \mathbb{N}$, if and only if $\mathsf{cosk}_{n+1}f : \mathsf{cosk}_{n+1}X \to \mathsf{cosk}_{n+1}Y$ is a weak equivalence for all $n \in \mathbb{N}$.

(vi) We have $X \cong \mathsf{lim}_{n\in\mathbb{N}} X[n] \simeq \mathsf{holim}_{n\in\mathbb{N}} X[n]$ and $X \cong \mathsf{lim}_{n\in\mathbb{N}} \mathsf{cosk}_{n+1}(X)$.

Proof. (i). We will not use the properties of $X[n]$ in the sequel; so we only give a reference for its properties. For this part, see Proposition 8.2 in [May92]. We show that the $(n+1)$-th coskeleton of X is a Kan complex: consider $m \in \mathbb{N}$ and a map $\Lambda^m_k \to \mathsf{cosk}_{n+1}(X)$ where Λ^m_k is the k-th horn of Δ^m, with $k \leq m$. Then if $m \leq n+1$, the map clearly extends to Δ^m because X is a Kan complex and the $n+1$ first simplicial levels of X and $\mathsf{cosk}_{n+1}(X)$ coincide. If $m > n+1$, we consider the adjoint map $\mathsf{sk}_{n+1}(\Lambda^m_k) \to X$, and we must find an extension to $\mathsf{sk}_{n+1}(\Delta^m) \to X$. But if $m > n+2$, $\mathsf{sk}_{n+1}(\Lambda^m_k) = \mathsf{sk}_{n+1}(\Delta^m)$. If $m = n+2$, $\mathsf{sk}_{n+1}(\Lambda^m_k) = \Lambda^{n+2}_k$ and $\mathsf{sk}_{n+1}(\Delta^m) = \partial\Delta^{n+2}$, but then an extension exists because X is a Kan complex (extend to Δ^{n+2} and restrict to $\partial\Delta^{n+2}$).

(ii). The first claim follows directly from the fact that their $n+1$ skeleton coincide, therefore so do the n first homotopy groups, and that all homotopy groups in dimension strictly greater than n are trivial. For the case of $X[n]$, the latter appears as Theorem 8.4 in [May92], and for $\mathsf{cosk}_{n+1}X$ it follows directly from the characterizing property of the $(n+1)$-th coskeleton: any map from $\partial\Delta^k$ to it extends (uniquely) to Δ^k for $k \geq n+2$ (see just below Definition 2.16).

(iii). This is Proposition 8.2 in [May92]. In the case of the coskeleton, the map has the right lifting property with respect to all horn inclusions except possibly in dimension $n+2$.

(iv). As stated in Definition 2.16, coskeleton functors are right adjoints and therefore preserve limits. The fact that both constructions preserve weak equivalences follows directly by the properties of their homotopy groups in part (i). Preservation of homotopy limits is a formal consequence of the preservation of limits and weak equivalences, at least for model categories. We will do the argument in our context to fix ideas but it can easily be generalized. Indeed, let us view $_{\mathsf{S}}\mathsf{Set}$ as a homotopical category (so we keep the weak equivalences but forget about fibrations and cofibrations). Then $\mathsf{cosk}_n : {_{\mathsf{S}}\mathsf{Set}} \to {_{\mathsf{S}}\mathsf{Set}}$ preserves weak equivalences, it is a homotopical functor. In particular, it is a (left or right) derived functor for itself, and it is (left and right) deformable. The limit functor $\mathsf{Fun}(\mathcal{D}, {_{\mathsf{S}}\mathsf{Set}}) \to {_{\mathsf{S}}\mathsf{Set}}$ for $\mathcal{D}$ a small category is right deformable (fibrant replacement yields a deformation in the case of model categories). Therefore, if $\mathbb{R}$ denotes right derivation, we can compute:

$$\begin{aligned}\mathsf{holim}_{\mathcal{D}} \circ \mathsf{cosk}_{n+1} &\simeq \mathbb{R}(\mathsf{lim}_{\mathcal{D}}) \circ \mathbb{R}(\mathsf{cosk}_{n+1}) \simeq \mathbb{R}(\mathsf{lim}_{\mathcal{D}} \circ \mathsf{cosk}_{n+1}) \simeq \mathbb{R}(\mathsf{cosk}_{n+1} \circ \mathsf{lim}_{\mathcal{D}}) \\ &\simeq \mathbb{R}(\mathsf{cosk}_{n+1}) \circ \mathbb{R}(\mathsf{lim}_{\mathcal{D}}) \simeq \mathsf{cosk}_{n+1} \circ \mathsf{holim}_{\mathcal{D}}\end{aligned}$$

because for deformable functors in adjunctions, derivation commutes with composition (this is Corollary 44.5 in [DHKS04]).

(v). The "only if" direction is given by part (iv). For the converse, we just notice using the second bullet point that if for some $n \in \mathbb{N}$, the map $f[n]$, respectively $\mathsf{cosk}_{n+1}(f)$, induces isomorphisms on all homotopy groups, then f induces isomorphisms on all homotopy groups in dimensions at most n. Since n is arbitrary, we deduce the statement.

(vi). This follows from the fact that limits are computed levelwise, and for all $k \in \mathbb{N}$, $X[m]$ and $\mathsf{cosk}_{m+1}(X)$ have the same k-simplices as X for all $k \leq m+1$. The equivalence with the homotopy limit comes from the fact that a tower of fibrations and fibrant objects is a Reedy fibrant diagram, in the injective model structure (see the page "Homotopy limit" in the nLab, Section 7, paragraph "Sequential homotopy (co)limits"), and thus the homotopy limit is a strict limit in this situation.

□

The essential features of the coskeleton construction for us will be the preservation of fibrant objects, weak equivalences and homotopy limits; but we will unfortunately not directly obtain a tower of fibrations.

5.3 In the motivic world

Having now gathered the topological intuition and the technical simplicial background, we can finally build the advertised towers in $\mathsf{Spc}_{\mathbb{A}^1}$.

5.3.1 Existence of Postnikov towers

The main goal of this subsection is to give a proof of Theorem 6.1 in [AF14], which is presented as a "collage" of results from various references. Here is the statement:

Theorem 5.7 (Existence of Postnikov towers in $\mathsf{Spc}_{\mathbb{A}^1,*}$)**.** *Let $Y \in \mathsf{Spc}_{\mathbb{A}^1,*}$ with $\pi_0^{\mathbb{A}^1}(Y) = *$ (namely Y is $\mathbb{A}^1$-connected). Then there exists a tower of $\mathbb{A}^1$-connected fibrant objects of $\mathsf{Spc}_{\mathbb{A}^1,*}$:*

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow f_2 \\ & & Y[2] \\ & \nearrow^{p_2} & \downarrow f_1 \\ & \nearrow^{p_1} & Y[1] \\ & & \downarrow f_0 \\ Y & \xrightarrow{p_0} & * \end{array}$$

with the following properties:

(i) (Homotopy groups:) The morphism p_i induces an isomorphism $\pi_j^{\mathbb{A}^1}(Y) \to \pi_j^{\mathbb{A}^1}(Y[i])$ for all $j \leq i$ and $\pi_j^{\mathbb{A}^1}(Y[i]) = 0$ for all $j > i$.

(ii) (Fibrations:) All morphisms f_i are fibrations in $\mathsf{Spc}_{\mathbb{A}^1,*}$ with homotopy fiber a $K(\pi_{i+1}^{\mathbb{A}^1}(Y), i+1)$, by which we mean that the homotopy fiber has a single non trivial homotopy sheaf, given by $\pi_{i+1}^{\mathbb{A}^1}(Y)$ in dimension $i+1$.

(iii) (Convergence:) The map $Y \to \mathsf{holim}_i\ Y[i] \simeq \mathsf{lim}_i\ Y[i]$ induced by the maps $\{p_i\}$ is an $\mathbb{A}^1$-weak equivalence.

(iv) (Equivariance:) The morphism f_i is a twisted principal fibration*, in the sense that there is up to $\mathbb{A}^1$-homotopy a unique morphism k_i called the* i-th k-invariant *such that the following square is a homotopy pullback in $\mathsf{Spc}_{\mathbb{A}^1,*}$:*

$$\begin{array}{ccc} Y[i+1] & \longrightarrow & B\pi_1^{\mathbb{A}^1}(Y) \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ Y[i] & \xrightarrow{k_i} & K^{\pi_1^{\mathbb{A}^1}(Y)}(\pi_{i+1}^{\mathbb{A}^1}(Y), i+2) \end{array}$$

with

$$K^{\pi_1^{\mathbb{A}^1}(Y)}(\pi_{i+1}^{\mathbb{A}^1}(Y), i+2) = E(\pi_1^{\mathbb{A}^1}(Y)) \times_{\pi_1^{\mathbb{A}^1}(Y)} K(\pi_{i+1}^{\mathbb{A}^1}(Y), i+2)$$

the twisted Eilenberg-MacLane object, *with* $\pi_1^{\mathbb{A}^1}(Y)$ *acting on* $\pi_{i+1}^{\mathbb{A}^1}(Y)$ *objectwise as usual for simplicial sets, viewing them as the simplicial homotopy sheaves for* $L_{\mathbb{A}^1} L_{\mathsf{Nis}} Y$.

We will not prove item (iv).

In [MV99], the objects $Y[i]$ are constructed as follows:

Definition 5.8 (Morel-Voevodsky motivic Postnikov tower)**.** Given $Y \in \mathsf{Spc}_{\mathbb{A}^1,*}$, let $Y[0] = *$ and for all $i \in \mathbb{N}^*$, we define the i*-th Postnikov truncation of* Y as the simplicial presheaf $Y[i] \in \mathsf{Spc}_{\mathbb{A}^1,*}$ given by the (Nisnevich) sheafification of the presheaf $U \in \mathsf{Sm}_S \mapsto Y(U)[i]$, where $Y(U)[i]$ is as in Definition 5.5 for $Y(U) \in {}_S\mathsf{Set}$.

In [AF14], the three main references cited for this theorem are [GJ09] (in the case of simplicial sets), [MV99] and [Mor12] (using a model structure on simplicial presheaves defined in a different way than the one we use, although they are closely related (see Subsection 6.1)). The main difficulties we faced in reconstructing a proof of Theorem 5.7 are that the homotopy presheaves in these references are computed objectwise or at stalks; and the model structure used is closely related to Spc instead of $\mathsf{Spc}_{\mathbb{A}^1}$. This is why we will not use the construction of Definition 5.8, which corresponds to Definition 5.5 in the simplicial setting, but a slightly different one, corresponding to the coskeleton construction for simplicial sets. Yet another, more abstract, construction is described in the Appendix in Remark A.7, using other model structures on ${}_S\mathsf{Pre}(Sm_S)$ whose fibrant objects are n-truncated.

To compute homotopy sheaves, it is desirable to work directly with $\mathbb{A}^1$-local objects, without having to take a fibrant replacement. This is the main reason for our choice of using the coskeleton construction of the Postnikov sections instead of the construction of Definition 5.5. Preservation of Kan complexes, weak equivalences and homotopy limits in ${}_S\mathsf{Set}$ by the coskeleton functors permits the preservation of the different notions of fibrancy for presheaves (and of sheaves, since it preserves the equalizer and products appearing in the definition of a sheaf). Since this will not directly yield a tower of fibrations, we will have to proceed to a replacement; even if it makes the construction less explicit. On the other hand, the construction presented for instance in [MV99] or [Mor12] requires sheafification, which is probably more explicit but still not straightforward. It does not preserve the different notions of fibrant objects, at least in the construction of $\mathsf{Spc}_{\mathbb{A}^1}$ we described; thus it makes it difficult to compute the homotopy sheaves of the candidate Postnikov truncations.

Proof of Theorem 5.7. Firstly, note that it suffices to prove the statement for Y fibrant in $\mathsf{Spc}_{\mathbb{A}^1,*}$. Indeed, contrarily to the CW-approximation in the topological case, the fibrant replacement is done on the right. Namely, we can consider the $\mathbb{A}^1$-weak equivalence $\ell : Y \xrightarrow{\sim} L_{\mathbb{A}^1} L_{\mathsf{Nis}}(Y)$, apply Theorem 5.7 to the localization of Y, and replace the collection $\{p_i\}_{i\geq 0}$ by $\{p_i \circ \ell\}_{i\geq 0}$. Since $\mathbb{A}^1$-equivalences induce isomorphisms on homotopy sheaves (Proposition 4.3), and since homotopy limits are invariant under weak equivalence, the tower obtained in this way still has the desired properties.

Our first candidate for the tower is defined by $Y[0] = *$ and $Y[i] = \mathsf{cosk}_{i+1}(Y)$ for all $i \geq 1$, where the coskeleton functor is applied objectwise; it also comes with a canonical basepoint if Y is pointed. For all $i \geq 1$, the map $p_i : Y \to Y[i]$ is obtained objectwise using the natural transformation $\mathsf{id}_{{}_S\mathsf{Set}} \to \mathsf{cosk}_{i+1}$. And $f_i : Y[i+1] \to Y[i]$ is obtained objectwise in the following way: for any simplicial set X, we have by adjunction

$${}_S\mathsf{Set}(\mathsf{cosk}_{i+2}(X), \mathsf{cosk}_{i+1}(X)) \cong {}_S\mathsf{Set}(\mathsf{sk}_{i+1}(\mathsf{cosk}_{i+2}(X)), X) \cong {}_S\mathsf{Set}(\mathsf{sk}_{i+1}(X), X)$$

and in the latter hom-set we have the natural map given by the adjunction. The maps f_i might not be fibrations, but we will solve this problem later in the same way as in the topological case.

Homotopy groups: To compute the homotopy groups of our candidate truncations, we first check fibrancy using the properties given in Lemma 5.6. Fix $n \geq 1$. For all $U \in \mathsf{Sm}_S$, $Y(U)$ is by hypothesis a Kan complex, therefore so is $\mathsf{cosk}_n(Y(U))$. In the case where S is Noetherian of finite Krull dimension, to check Nisnevich fibrancy we use Theorem 2.34. We have $Y(\emptyset) \simeq *$ so $\mathrm{cosk}_n(Y(\emptyset)) \simeq \mathrm{cosk}_n(*) = *$ by preservation of weak equivalences. Moreover, given any elementary Nisnevich square formed by maps $U \to X$ and $V \to X$, we have $Y(X) \simeq Y(U) \times^h_{Y(U \times_X V)} Y(V)$ since Y is Nisnevich fibrant. Since cosk_n preserves homotopy pushouts, we have the same weak equivalence with Y replaced by $\mathrm{cosk}_n(Y)$. This shows that $\mathrm{cosk}_n(Y)$ is Nisnevich fibrant. If we have no assumptions on S, one can instead check the full Nisnevich descent condition, in the exact same way. For any $U \in \mathrm{Sm}_S$, since cosk_n preserves weak equivalences and Y is $\mathbb{A}^1$-local, the projection induces a weak equivalence $\mathrm{cosk}_n(Y(U)) \to \mathrm{cosk}_n(Y(U \times_S \mathbb{A}^1))$. Therefore, $\mathrm{cosk}_n Y$ is $\mathbb{A}^1$-local (Remark 2.44).

Therefore, since both Y and $\mathrm{cosk}_n Y$ are $\mathbb{A}^1$-local, their m-th $\mathbb{A}^1$-homotopy sheaves can be computed as the sheaves associated with the presheaves mapping $U \in \mathrm{Sm}_S$ to $\pi_m(Y(U))$, respectively $\pi_m(\mathrm{cosk}_n(Y(U)))$, for the basepoints induced by that of Y (as we saw in Remark 4.2). In view of part (ii) in Lemma 5.6, this directly implies the conditions we ask for on $\mathbb{A}^1$-homotopy sheaves.

Fibrations: To replace the maps f_i with fibrations (in $\mathrm{Spc}_{\mathbb{A}^1}$), we proceed in the exact same way as in Step 2 of the proof of Theorem 5.1, by factoring the maps inductively, starting at the bottom of the tower, into acyclic cofibrations followed by fibrations. This operation preserves fibrancy of the objects in the tower; but now the maps between them have been replaced by fibrations. We abuse notation and keep the same names for the stages of this new tower, and the maps involved. The statement about the homotopy groups of the fibers of the $\{f_i\}$ follows in the same way as in the topological case, using the long exact sequence of Proposition 4.3, part (iii).

Convergence: Since we are considering a tower of fibrant objects and fibrations, it is a general fact that the homotopy limit is a strict limit (we already saw this in the proof for the topological case). So we only prove the claim about the homotopy limit.

The method used in the topological case might not work this time. Indeed, although the topological statement about the homotopy groups of a limit of a tower of fibrations has a version applying to simplicial sets, we do not have fibrations in the coskeleton tower, whereas in the tower where we replaced the maps by fibrations we do not know the objectwise homotopy groups.

So we have to use another trick. Since for all $n \geq 1$, $\mathrm{cosk}_{n+1} Y$ is $\mathbb{A}^1$-weakly equivalent to Y[n] by construction, the homotopy limit can also be computed as $\mathrm{holim}_n \, \mathrm{cosk}_{n+1} Y$. Considering the fact that the identity $\mathrm{Spc}_{\mathbb{A}^1,*} \to {}_S\mathrm{Pre}(\mathrm{Sm}_S)_*$ is right Quillen, the homotopy limit computed in $\mathrm{Spc}_{\mathbb{A}^1,*}$ is objectwise weakly equivalent to the homotopy limit computed in ${}_S\mathrm{Pre}(\mathrm{Sm}_S)_*$, in particular they are also $\mathbb{A}^1$-equivalent. Thus if we show that Y is objectwise weakly equivalent to the homotopy limit computed in ${}_S\mathrm{Pre}(\mathrm{Sm}_S)_*$, we are done. Since the latter category has the projective model structure, homotopy limits are computed objectwise. Let $U \in \mathrm{Sm}_S$ be a scheme. Using parts (iv) and (v) in Lemma 5.6 above, it suffices to show that $\mathrm{cosk}_m(Y(U)) \to \mathrm{holim}_n \, \mathrm{cosk}_m(\mathrm{cosk}_{n+1}(Y(U)))$ is a weak equivalence in ${}_S\mathrm{Set}$ for all $m \geq 1$. Note that $\mathrm{cosk}_m \circ \mathrm{cosk}_n = \mathrm{cosk}_{\min\{m,n\}}$, and therefore the homotopy limit is just that of the tower

$$\cdots \mathrm{cosk}_m Y(U) \to \cdots \to \mathrm{cosk}_m Y(U) \to \mathrm{cosk}_{m-1} Y(U) \to \cdots \to \mathrm{cosk}_1 Y(U).$$

By a formal argument, the (homotopy) limit of a tower is not affected by forgetting finitely many terms at the beginning, so in our case the homotopy limit becomes $\mathrm{holim}_n \, \mathrm{cosk}_m Y(U)$, which is just $\mathrm{cosk}_m Y(U)$. The induced map from $\mathrm{cosk}_m Y(U)$ is clearly the identity, and thus it is a weak equivalence. As explained above, this suffices to conclude. □

5.3.2 Obstruction theory and vector bundles

Similarly as in the topological case, a map $Z \to Y[i]$ for some pointed simplicial presheaf Z lifts to $Y[i+1]$ if and only if the image of the k-invariant k_i vanishes in $[Z, K^{\pi_1^{\mathbb{A}^1}(Y)}(\pi_{i+1}^{\mathbb{A}^1}(Y), i+2)]_{\mathbb{A}^1,*}$. With these methods, the material presented in Appendix B in [Mor12] allows one to show that Postnikov towers behave like approximations in limited dimension:

Proposition 5.9 (Approximation by Postnikov truncations (Proposition 6.2 in [AF14])). *Let X be a smooth variety of dimension d over a field k. Then for any $Y \in \mathrm{Spc}_{\mathbb{A}^1,*}$ with $\pi_0^{\mathbb{A}^1}(Y) = *$ and for all $i \geq d-1$, the morphism $p_i : Y \to Y[i]$ induces a surjection, if $i \geq d$ even a bijection:*

$$[X, Y]_{\mathbb{A}^1} \to [X, Y[i]]_{\mathbb{A}^1}.$$

Idea of a proof. The properties we will cite are proved in [Mor12], Appendix B. For surjectivity, starting with a map $X \to Y[i]$ (maps in the homotopy category are in this situation represented by homotopy classes of maps since $Y[i]$ is $\mathbb{A}^1$-fibrant by construction and X is cofibrant as it is a representable object), we want to lift it to the next truncations of Y. By convergence of the tower, we know that a lift to Y exists if the k-invariants vanish in all dimensions greater than i. The key input is to show that twisted Eilenberg-MacLane spaces also represent some cohomology groups, with different coefficients but in the same dimension as in the usual case. When $i \geq d-1$, we have $i+2 > \mathsf{dim}(X)$, and therefore sheaf cohomology on X is always trivial in dimensions at least $i+2$. So the suitable k-invariants vanish. For injectivity, one shows that the space of lifts up to homotopy is a quotient of some sheaf cohomology group on X in one dimension less, so if $i \geq d$ this is sufficient for the corresponding cohomology group to vanish. □

The following theorem, which we will not prove, gives a consequence almost purely in algebraic geometry of the previous motivic discussion:

Theorem 5.10 (Trivial summands for high dimensional bundles (Proposition 8.9 in [AE17])). *Let X be a smooth affine variety over a field k. If E is a vector bundle over X with $n = \mathsf{rank}(E) > \mathsf{dim}(X) = d$, then E splits off a trivial direct summand, i.e. it is $\mathbb{A}^1$-equivalent to $E' \oplus \epsilon$ with ϵ a trivial bundle of rank $n-d$ over X.*

Ingredients of one proof. One can shows that vector bundles of rank n over an affine base are represented by the classifying space $B(GL_n)$ of the group scheme GL_n. The latter is the closed subscheme of $\mathbb{A}^{n^2+1}$ defined by the equations imposing the determinant to be invertible (the first n^2 parameters are thought of as the coefficients of a matrix and the extra parameter represents the determinant). Then, using Proposition 5.9, the Postnikov towers (and homotopy groups) of $B(GL_n)$ and $B(GL_d)$ are used to find a d-dimensional summand. □

Motivic Postnikov towers have been used in the proofs of other classification results for vector bundles (see for instance [AF14] and other articles by Asok and Fasel). They can also be constructed in the *stable* motivic homotopy category, together with other kinds of towers approximating a given motivic spectrum. We will say some words about this in Section 6.

6 Related topics

This section is more descriptive and contains a collection of topics that could constitute extensions to this project.

6.1 Other models for a motivic category

Morel and Voevodsky originally used in [MV99] a model structure on ${}_S\mathsf{Pre}(Sm_S)$ which is a bit different from the one we studied, namely $\mathsf{Spc}_{\mathbb{A}^1}$. The following definition requires the notions of points and stalks for a site; the necessary background can be found in subsection A.4 in the Appendix.

Definition 6.1 (The Jardine-Joyal model structure ([Jar87a] for presheaves, Joyal for sheaves))**.** Let $(\mathcal{C}, \tau)$ be a site with enough points. There is a model structure on the category of (pre)sheaves over $\mathcal{C}$ such that:

- the weak equivalences are the *topological weak equivalences*, namely the morphisms of (pre)sheaves $f : X \to Y$ such that for all $U \in \mathcal{C}$ and $x \in X(U)_0$, f induces an isomorphism between the sheaves on $\mathcal{C} \downarrow U$ associated with the presheaves $(V \to U) \mapsto \pi_n(X(V), x_V)$ and $(V \to U) \mapsto \pi_n(Y(V), f(x_V))$, where x_V denotes the image of x by the corresponding map $X(U) \to X(V)$.
- these weak equivalences can be described as the *combinatorial weak equivalences* when the domain and codomain are locally fibrant objects (namely (pre)sheaves Z such that for any point u of the site, the stalks Z_u is a Kan complex). These can be defined as the morphisms $f : X \to Y$ of (pre)sheaves such that for any point u of the site, the induced map $f_u : X_u \to Y_u$ is a weak equivalence of simplicial sets.
- the cofibrations are defined objectwise (objectwise monomorphisms).
- the fibrations are defined by the right lifting property, and are called *global fibrations*.

The definition of topological weak equivalences highlights a subtlety about basepoints: the vertices $x \in X(U)_0$ under consideration are not global basepoints, in the sense that they do not necessarily come from the restriction of a basepoint for X as a simplicial presheaf.

The Jardine model structure is morally closer to the injective model structure, whereas Spc was closer to the projective one. Its weak equivalences can be qualified as "stalkwise weak equivalences". Morel and Voevodsky use the version with sheaves and not presheaves, and proceed to a Bousfield localization with respect to all projections $X \times_S \mathbb{A}^1 \to X$ for $X \in \mathsf{Sm}_S$. Yet another construction is chosen by [AHW17], it is the same as the construction we presented in Section 2, but instead of Bousfield-localizing with respect to all Nisnevich *hypercovers*, they only ask for descent with respect to *coverings* in the Nisnevich topology. A discussion on the differences between descent and hyperdescent in the context of topoi can be found in Section 6.5.4 in [Lur09].

Here are three comparison theorems to understand the relations between these model structures. We assume that our base scheme S is Noetherian of finite Krull dimension.

Theorem 6.2 (Spc and the Jardine model structure (Section 2 and Theorem 6.2 in [DHI04]))**.**

(i) The model structure on $L_{\mathsf{Nis}}({}_S\mathsf{Pre}(\mathsf{Sm}_S))$ *is the same as the left Bousfield localization of* ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$ *(with its projective model structure) with respect to the class of topological weak equivalences.*

(ii) The same statement holds for the injective model structures, in which case we obtain exactly the Jardine model structure from Definition 6.1.

(iii) The projective and injective constructions are Quillen equivalent.

Theorem 6.3 ($\mathsf{Spc}_{\mathbb{A}^1}$ and a localization of the Jardine model structure (Theorem 2.17 in [DRØ03])**.** *The identity on the underlying categories defines a left Quillen functor from* $\mathsf{Spc}_{\mathbb{A}^1}$ *to the left Bousfield localization of the Jardine model structure with respect to the the zero section* $S \to \mathbb{A}^1$ *(which we will call the* $\mathbb{A}^1$*-local Jardine model structure). Moreover, the weak equivalences in the two categories coincide.*

A last comparison result is proved in the Appendix (Theorem A.5):

Theorem 6.4 (Hyperdescent vs descent for bounded-above presheaves (Corollary A.9 in [DHI04])). *Let* $n \in \mathbb{N}^*$. *Let* $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ *be presheaf of Kan complexes, such that for all* $U \in \mathsf{Sm}_S$, $\pi_m(X(U))$ *is trivial for any choice of basepoint and for all* $m > n$. *Then* X *satisfies Nisnevich descent if and only if it satisfies descent with respect to Čech hypercovers.*

6.2 The point of view of universal homotopy theories

In the paper [Dug01b], Dugger looks into the question of associating to any (small) category a model category in a "universal way", in a "generators and relations" approach. To illustrate this philosophy, consider the homotopy theory of CW-complexes for instance. It is characterized by the fact that $\mathsf{hocolim}\ U_\bullet \to X$ and $X \times I \to X$ are weak equivalences for any Cech complex $U_\bullet$ representing a covering of topological space X. With the approach of Dugger, the universal model category built from Top by imposing these relations recovers the usual homotopy theory for CW-complexes. The motivic category $\mathsf{Spc}_{\mathbb{A}^1}$ can also be seen as a universal model category in this way. When $S = \mathsf{Spec}(k)$ is a field which embeds in $\mathbb{C}$, these ideas allows Dugger to construct in a "simple" way *topological realization functors* from $\mathsf{Spc}_{\mathbb{A}^1}$ to the category of topological spaces, with restricts to analytification on the image of Sm_S in $\mathsf{Spc}_{\mathbb{A}^1}$ (basically, analytification makes the set of complex points of a scheme into a complex analytic space). The same is true for the category built in the same way as $\mathsf{Spc}_{\mathbb{A}^1}$ but considering the étale Grothendieck topology on Sm_S. A very interesting intuitive discussion about all this can be found in the introduction to [Dug98]. We state some results to understand the relevance of universal homotopy theories in relation to our context:

Proposition 6.5 (Property of the "universal homotopy theory" (Proposition 1.1 in [Dug01b]). *Let* $\mathcal{C}$ *be a small category. Let* $U\mathcal{C} := {}_S\mathsf{Pre}(\mathcal{C})$ *be the model category of simplicial presheaves on* $\mathcal{C}$ *with the projective model structure. Then, the Yoneda embedding* $\iota : \mathcal{C} \to U\mathcal{C}$ *has the property that any map* $\mathcal{C} \to \mathcal{M}$ *for* $\mathcal{M}$ *a model category factors through* $U\mathcal{C}$*, and the category of such factorizations is contractible.*

Here is a definition of presentations by generators and relations for model categories:

Definition 6.6 (Small presentation of a category (Definition 6.1 in [Dug01b])). If $\mathcal{M}$ is a model category, a *small presentation* of $\mathcal{M}$ is the data of a small category $\mathcal{C}$, a Quillen pair $\mathsf{Re} : U\mathcal{C} \rightleftarrows M : \mathsf{Sing}$ and a set S of maps in $U\mathcal{C}$, such that the left derived functor of Re takes maps in S to weak equivalences and the induced Quillen pair from the left Bousfield localisation of $U\mathcal{C}$ with respect to S to the category $\mathcal{M}$ is a Quillen equivalence.

Theorem 6.7 (Existence of presentations (Theorem 6.3 in [Dug01b])). *Any combinatorial model category admits a small presentation.*

Theorem 6.8 (The motivic category as a universal homotopy theory (Theorem 8.1 in [Dug01b])). *Let* k *be a field. Consider* $U(\mathsf{Sm}_k)_{\mathbb{A}^1}$ *the universal model category on* Sm_k*, with the relations that Nisnevich hypercovers and projections with respect to the affine line are weak equivalences. Then there is a Quillen equivalence from* $U(\mathsf{Sm}_k)_{\mathbb{A}^1}$ *to the motivic category of Morel and Voevodsky defined in [MV99].*

Note that $U(\mathsf{Sm}_k)_{\mathbb{A}^1}$ is nothing but $\mathsf{Spc}_{\mathbb{A}^1}$. Therefore, this statement gives yet another variation of the comparison theorems in subsection 6.1.

6.3 The stable motivic homotopy category

We view this subsection as a way to introduce more "nice words" and ideas of subjects to explore to go further. It is not by any means precise or rigorous. Most of it is inspired by the talk "Motivic Cohomology: past, present and future" by Marc Levine ([Lev22]).

The methods leading to the stable homotopy category for topological spaces can be applied to the motivic setting to define a stable motivic homotopy category. Let us first recall part of the story in the topological case (our reference is [Mal23]), and then we will do the parallel with the motivic world. For topological spaces, one first defines a category of spectra, namely pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$ with structure maps $\Sigma X_n \cong \mathbb{S}^1 \wedge X_n \to X_{n+1}$ for all $n \in \mathbb{N}$. Every topological space X gives rise to a spectra by taking its *infinite suspension spectrum* $\Sigma^\infty(X) = \{\Sigma^n(X_+)\}$ where Σ is the (reduced) suspension, or equivalently the smash product with $\mathbb{S}^1$. In an attempt to define a appropriate smash product for spectra, one defines the category of symmetric spectra. The latter admits a smash product making it into a symmetric monoidal category. One then defines the stable homotopy groups of spectra, and weak equivalences as the morphisms of spectra that induce isomorphisms on all stable homotopy groups. The corresponding homotopy category is called the stable homotopy category, and comes with a one-parameter family of *invertible* suspension functors $\{\Sigma^n\}_{n\in\mathbb{N}} = \{\Sigma^\infty(S^n) \wedge -\}_{n\in\mathbb{N}}$. This category of spectra admits Spanier-Whitehead duality: this means that dual objects (in the monoidal sense) exist for all finite CW-spectra for example ([SW55], Section 4.2.2 in [Mal23]). Moreover, the stable homotopy category becomes triangulated.

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To reproduce these constructions in the motivic setting, one idea would be to use the actual categorical suspension functor, which is given by the smash product with the simplicial circle $\mathcal{S}^1$, as we saw in Proposition 3.7. The corresponding category of spectra can be defined ([VROsr07], Section 2.2), but it turns out that suspension with respect to $\mathbb{P}^1$ (which is the smash product of the two circles) is more appropriate. Then one defines a motivic spectra as a sequence $\{X_n\}_{n\in\mathbb{N}}$ of objects of $\mathsf{Spc}_{\mathbb{A}^1,*}$ with structure maps $\mathbb{P}^1 \wedge X_n \to X_{n+1}$. Then, any motivic space gives rise to a spectrum by taking its infinite suspension spectrum. Once more, the stable motivic category, or rather its "symmetric spectra" version, can be endowed with a smash product making it into a symmetric monoidal category. Defining and inverting the suitable weak equivalences yields the stable motivic homotopy category, which now comes with a bigraded family of *invertible* suspension operators. The choice of $\mathbb{P}^1$ can be partly justified by the fact that it allows us to have Spanier-Whitehead duality in this context as well, whereas suspension with respect to $\mathcal{S}^1$ does not ([Voe03], [Rio05]). The stable motivic homotopy category is also triangulated, and is a nice place to study bigraded cohomology theories for smooth schemes, just as spectra of topological spaces can be used to study generalized cohomology theories.

Many interesting objects also live in the stable motivic category, for example a spectrum KGL which represents algebraic K-theory ([MV99], Section 4.3)(algebraic K-theory is also representable in the unstable motivic category), or a motivic spectrum KQ representing Hermitian K-theory. Motivic cohomology, which is some version of singular cohomology for schemes, is also representable by a motivic spectrum. All of this is closely related to the category of Voevodsky's motives, a candidate for the derived category of Grothendieck's hypothetical category of motives, in a certain sense. The construction is very related to the steps leading to $\mathsf{Spc}_{\mathbb{A}^1}$. Indeed, one first defines a category of "presheaves with transfers" over some category of "correspondences", whose objects are the same as those of Sm_S. Each such presheaf yields an underlying presheaf on Sm_S, and then one requires a condition similar to Nisnevich hyperdescent and $\mathbb{A}^1$-homotopy invariance for these presheaves.

In the stable motivic category, various kinds of towers approximating a given object can be defined (this part is inspired by the talk "Hermitian K-groups and motives" by Paul Arne Østvær ([Øst24])). This also yields filtrations of the stable motivic category itself. Postnikov towers can be defined in this setting, but also, in the same way as a spectrum of topological spaces can be approximated by a tower of more and more connective covers, for a motivic spectrum one defines its effective (respectively very effective) *slice filtration*. This is a tower of more and more connective covers with respect to the Tate circle (respectively, to $\mathbb{P}^1$), although we will not define precisely what this means. The cofibers of the successive maps in the tower are called the *slices* of the motivic spectrum under consideration. These slices are in general quite hard to compute; known examples include the 0th slice of the sphere spectrum $\Sigma^\infty(\mathcal{S}^0)$ (see [Voe04]), the effective slices of KGL, and the effective and very effective slices of KQ (for results and references, see [Øst24]). There is a topological realization functor from the stable motivic homotopy category to the stable homotopy category, philosophically it corresponds to the taking the complex points of a scheme. Under some conditions, this realization functor maps slice towers to the even stages of a Whitehead tower ([GRSØ12], paragraph 3.3). The stable motivic category and stable motivic homotopy category have a wealth of other good properties and interesting objects; for instance one can recover some classical spectral sequences from this motivic setting.

7 Conclusion

The main goal of this semester project was to understand in some detail one construction of a motivic category, and to become quite familiar with the basic objects and methods of $\mathbb{A}^1$-homotopy theory. We began by implementing the three-steps program outlined in the introduction for the construction of a motivic category. This allowed us to translate several important topological construction into the language of motivic spaces. In particular, we studied three different flavours of homotopy sheaves of groups and explored their relations.

Throughout this discussion, we have been playing with both the algebraic geometry and the topological information encoded in the $\mathbb{A}^1$-homotopy theory. On the technical side, we have been using a lot the machinery of model categories and Bousfield localizations, together with homotopy limits and colimits, repeatedly coming back and forth between several different model structures on the category of simplicial presheaves over smooth schemes. The ones that appeared the most often are the three stages of the construction from Section 2, namely the projective model structure, its Bousfield localization at Nisnevich hypercovers, and the further localization at projections with respect to $\mathbb{A}^1$. Later on, we also met the Jardine model structure, and various other ones in subsection A.2, while studying the relation between Čech descent and Nisnevich (hyper)descent.

This project covers only very few topics, but we hope the treatment we have given of these questions was reasonably detailed and precise. There is a wealth of interesting related subjects to study. The theory of simplicial presheaves on Grothendieck sites is already very rich. In our case, this would correspond to studying the second step in our construction, namely the localization at Nisnevich hypercovers, before $\mathbb{A}^1$-localization. Moreover, several different versions of a motivic category are available, as we saw in subsection 6.1. Navigating between the different points of view represented in the literature was certainly one of the main difficulties in this project. There is also room for variations in the choice of the "model" we are working with: our discussion was entirely based on model categories, but one can perform similar constructions with ∞-categories for instance, and this is often the more modern take on this subject.

More generally, other variants are possible: for instance, many things can also be realized with the same construction in the étale topology instead; there are also versions of a motivic category defined with non-necessarily smooth schemes, although the theory is for the moment less developed.

Motivic homotopy theory has demonstrated its usefulness for solving pre-existing problems, such as the Milnor and Bloch-Kato conjectures ([Voe00], [Voe11]). It can also be used to study classification of vector bundles as we briefly mentioned in subsection 5.3, or K-theory for instance, among other applications. It has become a very rich subject in its own right, with its proper beautiful theorems, open problems and conjectures.

With more time, we could have explored in greater details the notion of a twisted principal fibration and the obstruction theory we briefly discussed at the end of subsection 5.3, or any of the interesting topics listed in Section 6. To go further, Antieau and Elmanto also give several exercises in section 9 of their article [AE17]. For a survey of some recent developments and tendencies in motivic homotopy theory, we refer to the expository article by [Lev20] by Marc Levine.

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A Complements about the Nisnevich topology

A.1 Elementary Nisnevich squares and the Nisnevich topology

In this subsection, our goal is to prove Proposition 2.12, namely that the topology τ_{cd} generated by the cd-structure of elementary distinguished Nisnevich squares coincides with the Nisnevich topology τ_{Nis}. To do so, it suffices to show that the two topologies have the same sheaves, in view of the following statement:

Proposition A.1 (Sheaves determine the topology)**.** *Two Grothendieck topologies τ and τ' on the same category $\mathcal{C}$ are equal if and only if they have the same sheaves (of sets), i.e. a presheaf on $\mathcal{C}$ is a sheaf with respect to τ if and only if it is a sheaf with respect to τ'.*

Sketch of proof. This appears for example as Theorem 7.50.2 in the Stacks project, Tag 00ZP. Essentially, one proof goes as follows: we use the equivalent definition of Grothendieck topologies in terms of coverings sieves, which are subfunctors of representable presheaves. One shows that a subfunctor $S \subseteq \mathcal{C}(-, U)$ for $U \in \mathcal{C}$ is a covering sieve for a topology τ if and only if the inclusion $S \to \mathcal{C}(-, U)$ becomes an isomorphism after sheafification. Assuming that sheaves with respect to τ and τ' agree, so do the sheafifications in the two topologies, and hence their covering sieves are the same. Therefore the topologies are equal. The other direction is tautological. □

To apply this strategy, we have to understand the sheaves in the topology τ_{cd}.

Proposition A.2 (Sheaves in a topology generated by a cd-structure ([Voe10], Lemma 2.9))**.** *Given a complete cd-structure on a category $\mathcal{C}$ with an initial object $\emptyset$, a presheaf (of sets) $\mathcal{F}$ on $\mathcal{C}$ is a sheaf in the topology generated by the cd-structure if and only if $\mathcal{F}(\emptyset) = *$ and $\mathcal{F}$ carries any square in the cd-structure to a pullback square of sets.*

So we need to know that the elementary Nisnevich squares form a *complete* cd-structure. We will not use the precise definition of this adjective, but here is one sufficient condition:

Proposition A.3 (Criterion for completeness of cd-structures ([Voe10], Lemma 2.5))**.** *Let $\mathcal{C}$ be a category with an initial object $\emptyset$, such that any morphism with codomain $\emptyset$ is an isomorphism. Consider a cd-structure in $\mathcal{C}$, such that its distinguished squares are stable by pullback along maps with codomain the lower right corner of the square. Then this cd-structure is complete.*

The proofs of Propositions A.2 and A.3 are only based on the definitions and on the axioms of a Grothendieck topology. We will therefore not prove them, choosing to detail only the part which is specific to the Nisnevich topology, in the proof of Proposition A.4.

Using Proposition A.3, we obtain that the cd-structure formed by elementary Nisnevich squares is complete, essentially because both open immersions and étale maps are stable under arbitrary base change. Combining the above results, Proposition 2.12 will be proven once we show the following statement:

Proposition A.4 (Sheaves in τ_{Nis} and τ_{cd} agree ([MV99], Proposition 1.4))**.** *A (non-empty) presheaf of sets $\mathcal{F}$ on Sm_S is a sheaf with respect to τ_{Nis} if and only if $\mathcal{F}(\emptyset) = *$ and $\mathcal{F}$ carries elementary Nisnevich squares to pullback squares (i.e. if and only if it is a sheaf with respect to τ_{cd}).*

Proof. To begin with, assume that $\mathcal{F}$ is a non-empty sheaf with respect to τ_{Nis}. We first show that $\mathcal{F}(\emptyset) = *$. Pick $X \in \mathsf{Sm}_S$ with $\mathcal{F}(X) \neq \emptyset$. There is a Nisnevich covering $\{X \to X; \emptyset \to X\}$ of X, and therefore an equalizer:

$$\mathcal{F}(X) \hookrightarrow \mathcal{F}(X) \times \mathcal{F}(\emptyset) \rightrightarrows \mathcal{F}(\emptyset) \times \mathcal{F}(\emptyset) \times \mathcal{F}(X) \times \mathcal{F}(\emptyset)$$

with the first map injective. The existence of this first map implies $\mathcal{F}(\emptyset) \neq \emptyset$. Assume $f \in \mathcal{F}(X)$ has image $(f; a)$. By contradiction, if $\mathcal{F}(\emptyset) \neq *$, there exists $b \in \mathcal{F}(\emptyset)$ distinct from a. Then $(f; b)$ has the same image by the two following maps in the equalizer; but does not belong to the image of the first map, because this map is given by the identity on the first component and f can only have a single image.

Consider now an elementary Nisnevich square formed by an open immersion $\iota : B \to C$ and an étale map $p : D \to C$. It is in particular a Nisnevich cover, and then, since $\mathcal{F}$ is a sheaf with resepct to τ_{Nis}, we have an equalizer:

$$\mathcal{F}(C) \xrightarrow{\mathcal{F}(p)\times\mathcal{F}(\iota)} \mathcal{F}(B) \times \mathcal{F}(D) \rightrightarrows \mathcal{F}(B \times_C B) \times \mathcal{F}(B \times_C D) \times \mathcal{F}(D \times_C B) \times \mathcal{F}(D \times_C D)$$

and the first map is injective. To show that $\mathcal{F}$ carries this Nisnevich square to a pullback, we have to check that the following is an equalizer, with the first map injective:

$$\mathcal{F}(C) \xrightarrow{\mathcal{F}(p)\times\mathcal{F}(\iota)} \mathcal{F}(B) \times \mathcal{F}(D) \rightrightarrows \mathcal{F}(B \times_C D)$$

Therefore it suffices to show that having equal images by the two maps on the right hand side of the first equalizer is equivalent to satisfying the same condition for the second equalizer. Namely, if an element x in $\mathcal{F}(B) \times \mathcal{F}(C)$ has its images in $\mathcal{F}(B \times_C D)$ by the two maps of the diagram equal, then we must show that its images by the two different maps induces by the projections are also equal in $\mathcal{F}(B \times_C B)$, $\mathcal{F}(D \times_C B)$ and $\mathcal{F}(D \times_C D)$. For the first one, it follows from the fact that $B \times_C B = B$ since B is Zariski open in C, and for the second one it follows from the isomorphism $B \times_C D \cong D \times_C B$. Finally, recall that $D \amalg ((D \times_C B) \times_B (D \times_C B)) \to D \times_C D$ is an étale cover (as we have seen in the proof of Proposition 2.35). In particular, $\mathcal{F}(D \times_C D) \to \mathcal{F}(D \amalg ((D \times_C B) \times_B (D \times_C B)))$ is injective, and we only have to show that the image of x under the two different maps $\mathcal{F}(D) \times \mathcal{F}(B) \to \mathcal{F}(D \amalg ((D \times_C B) \times_B (D \times_C B)))$ under consideration are equal. Both maps factor through the corresponding maps $\mathcal{F}(D) \times \mathcal{F}(B) \to \mathcal{F}(D \amalg (D \times_C B)) \cong \mathcal{F}(D) \times \mathcal{F}(D \times_C B)$ (Zariski sheaves already carry disjoint unions to products), but by hypothesis the two maps with codomain $\mathcal{F}(D \times_C B)$ we are considering already equalize x. This shows that $\mathcal{F}$ is a sheaf with respect to τ_{cd}.

For the converse, let $\mathcal{F}$ be a sheaf with respect to τ_{cd} (equivalently, satisfying the conditions of Proposition A.2). We first prove that $\mathcal{F}(\coprod_{i\leq n} U_i) = \prod_{i\leq n} \mathcal{F}(U_i)$ for any $U_1, \ldots, U_n \in \mathrm{Sm}_S$. The inclusions of U_n and $\coprod_{i<n} U_i$ into $\coprod_{i\leq n} U_i$ form a Nisnevich square with empty fiber product; and therefore $\mathcal{F}(\coprod_{i\leq n} U_i) \cong \mathcal{F}(\coprod_{i<n} U_i) \times \mathcal{F}(U_n)$, using that $\mathcal{F}(\emptyset) = *$. We conclude by induction.

We will show that $\mathcal{F}$ induces a suitable equalizer for every Nisnevich cover by induction on the minimal length of the filtration in the definition of a Nisnevich covering. If $\{p_i : U_i \to X\}_{i\leq n}$ is a Nisnevich covering with filtration $\emptyset = Z_1 \subseteq Z_0 = X$, then $\coprod_{i\leq n} p_i^{-1}(X) = \coprod_{i\leq n} U_i \to X$ admits a section s. Then $\mathcal{F}(X) \to \prod_{i\leq n} \mathcal{F}(U_i) = \mathcal{F}(\coprod_{i\leq n} U_i)$ has a retraction and is in particular injective. The rest of the exactness follows from the above and has nothing to do with the particular setting of the Nisnevich topology.

Assume now that $\mathcal{F}$ gives a suitable equalizer for any Nisnevich covering admitting a filtration by $k+1$ closed subschemes. Take a Nisnevich covering $\{p_i : U_i \to X\}_{i\leq n}$ admitting a filtration $\emptyset = Z_{k+1} \subseteq \cdots \subseteq Z_0 = X$ of length $k+2$. Then, there exists a section $s_k : Z_k \to \coprod_{i\leq n} p_i^{-1}(Z_k)$ of the map induced by the p_i's. Since the base change of $\coprod_{i\leq n} p_i$ to Z_k is still étale, and its pre-composition by s_k is the identity on Z_k, which is in particular étale, we obtain that s_k is étale (by the cancellation theorem for properties of morphisms (see 10.1.19 in [Vak17])). In particular, it is open and thus we can form the closed complement V_k of $\mathrm{Im}(s_k)$ in $\coprod_{i\leq n} p_i^{-1}(Z_k)$. The latter disjoint union is closed in $\coprod_{i\leq n} U_i$ and thus $W_k := (\coprod_{i\leq n} U_i) \setminus V_k$ is open in $\coprod_{i\leq n} U_i$. In particular, the maps $\{X \setminus Z_k \to X, W_k \to X\}$ form an elementary Nisnevich square. Also, the family $\{U_i \times_X (X \setminus Z_k) \to X \setminus Z_k\}_{i\leq n}$ is a Nisnevich covering (by base change) admitting a strictly shorter filtration $\emptyset = Z_k \setminus Z_k \subseteq Z_{k-1} \setminus Z_k \subseteq \cdots \subseteq Z_0 \setminus Z_k = X \setminus Z_k$. Therefore, by hypothesis we have equalizers:

$$\mathcal{F}(X) \hookrightarrow \mathcal{F}(X \setminus Z_k) \times \mathcal{F}(W_k) \rightrightarrows \mathcal{F}((X \setminus Z_k) \times_X W_k)$$

$$\mathcal{F}(X \setminus Z_k) \hookrightarrow \prod_{i\leq n} \mathcal{F}(U_i \times_X (X \setminus Z_k)) \rightrightarrows \prod_{i,j\leq n} \mathcal{F}(U_i \times_X U_j \times_X (X \setminus Z_k))$$

The map $\mathcal{F}(X) \to \prod_{i\leq n} \mathcal{F}(U_i) = \mathcal{F}(\coprod_{i\leq n} U_i)$ is injective because its post-composition by the map $\mathcal{F}(\coprod_{i\leq n} U_i) \to \mathcal{F}(W_k) \times \mathcal{F}(X \setminus Z_k)$ is exactly the first map in the first equalizer above, which is injective. Indeed, $W_k \to X$ and $X \setminus Z_k \to X$ both factor through $\coprod_{i\leq n} U_i$ because of the existence of the section s_k. The rest of the exactness follows by elementary computations and has nothing to do with the particular setting of the Nisnevich topology, so we allow ourselves to skip it. $\square$

A.2 Sheaves up to homotopy and hypersheaves

The added difficulty in this section compared to the preceding one is essentially due to two factors: the fact that we now have to deal with homotopical information (simplicial presheaves instead of presheaves of sets, and *homotopy* limits) and the fact that we are considering hypercovers instead of simple covers.

Theorem A.5 (Hyperdescent vs descent for bounded-above presheaves (Corollary A.9 in [DHI04])). *Let $n \in \mathbb{N}^*$. Let $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ be presheaf of Kan complexes, such that for all $U \in \mathsf{Sm}_S$, $\pi_m(X(U))$ is trivial for any choice of basepoint and for all $m > n$. Then X satisfies Nisnevich descent if and only if it satisfies descent with respect to Čech hypercovers.*

Proof. By "descent for Čech covers", we mean the same condition with the homotopy limit as for Nisnevich hyperdescent, but only for those hypercovers that are a Čech complex. The argument in [DHI04] is already pretty detailed. We reproduce it here only for the sake of completeness and because it's a beautiful proof. We will also add some details about the small object argument that is used.

Step 1. We will need new model structures on the category of simplicial presheaves. For this proof only, denote by $\mathcal{S}$ the model category ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$.

On the one hand, let $\mathcal{S}_n$ be the Bousfield localization of $\mathcal{S}$ at the set of maps $\underline{\partial\Delta^m} \times U \to \underline{\Delta^m} \times U$ for all $m > n$, as U varies over representatives for isomorphism classes of smooth schemes in Sm_S. This localization exists by Proposition 2.27. Its fibrant objects are the simplicial presheaves of Kan complexes with no homotopy above dimension $n-1$. Indeed, an object X is local with respect to this set of maps if and only if it is fibrant in $\mathcal{S}$ (i.e. it is a presheaf of Kan complexes) and

$$\mathsf{map}(\underline{\partial\Delta^m} \times U, X) \simeq \mathsf{map}(\partial\Delta^m, X(U)) \to \mathsf{map}(\underline{\Delta^m} \times U, X) \simeq \mathsf{map}(\Delta^m, X(U))$$

is a weak equivalence of simplicial sets for all $U \in \mathsf{Sm}_S$. Looking at path components, this implies that the Kan complex $X(U)$ has no homotopy in dimension $m-1$. Since this holds for all $m > n$, $X(U)$ has no homotopy above dimension $n-1$.

We can also take a further left Bousfield localization of $\mathcal{S}_n$, which we call $\check{\mathcal{S}}_n$, with respect to (representatives for isomorphism classes) of hypercovers that are Čech complexes. Therefore, fibrant objects in $\check{\mathcal{S}}_n$ are the fibrant objects of $\mathcal{S}_n$ satisfying Čech descent (in the same way as in Lemma 2.32). Note that this model structure can also be viewed as the Bousfield localization $\check{\mathcal{S}}$ of $\mathcal{S}$ at all Čech hypercovers, further localized with respect to the same set of maps as the localization defining $\mathcal{S}_n$.

Step 2. We reduce the problem to a question about the properties of hypercovers. Let X be as in the statement. By definition and by the description we just gave, it is fibrant in $\check{\mathcal{S}}_n$. Let $U_\bullet \to V$ be a hypercover in ${}_S\mathsf{Pre}(\mathsf{Sm}_S)$. We want to show that $\mathsf{map}(V, X) \to \mathsf{map}(U_\bullet, X)$ is a weak equivalence of simplicial sets. Since X is local in $\check{\mathcal{S}}_n$, it suffices to show that $U_\bullet \to V$ is a weak equivalence in $\check{\mathcal{S}}_n$, or equivalently, that both $U_\bullet \to \mathsf{cosk}_n U_\bullet$ and $\mathsf{cosk}_n U_\bullet \to V$ are weak equivalences in $\check{\mathcal{S}}_n$.

Step 3. We begin with $U_\bullet \to \mathsf{cosk}_n U_\bullet$. We claim that it is already a weak equivalence in $\mathcal{S}_n$. It suffices to show that we have an objectwise weak equivalence between fibrant replacements for $U_\bullet$ and $\mathsf{cosk}_n U_\bullet$ in $\mathcal{S}_n$. We claim that we can find a fibrant replacement that is an isomorphism on the first n simplicial levels. In particular, since $U_\bullet \to \mathsf{cosk}_n U_\bullet$ induces objectwise isomorphisms on the first $n-1$ homotopy groups (Proposition 5.6), and their fibrant replacements do not have any non-trivial objectwise homotopy groups above dimension $n-1$, the map induced between these fibrant replacements is an objectwise weak equivalence, as desired. So we are left to prove the claim. In a Bousfield localization of a model category satisfying the hypotheses of Proposition 2.27, fibrant replacements can be constructed using the small object argument ([Lur09], Proposition A.1.2.5). Let $Y \in \mathcal{S}_n$ be a fibrant object in $\mathcal{S}$, then the construction of a fibrant replacement in $\mathcal{S}_n$ by the small object argument goes as follows: there is some ordinal number κ, such that $Y \to RY$ is a weak equivalence in $\mathcal{S}_n$ and RY is fibrant in $\mathcal{S}_n$, with $RY = \mathsf{hocolim}_{\alpha<\kappa} Y_\alpha$. The object Y_α is constructed by transfinite induction: let $Y_0 = Y$, and given Y_α, we construct $Y_{\alpha+1}$ as the homotopy pushout on the left hand side:

$$\begin{array}{ccc} \coprod_{m>n, i\in I} \underline{\partial\Delta^m} \times A_i & \longrightarrow & Y_\alpha \\ \downarrow & & \downarrow \\ \coprod_{m>n, i\in I} \underline{\Delta^m} \times A_i & \longrightarrow & Y_{\alpha+1} \end{array} \qquad\qquad \begin{array}{ccc} \underline{\partial\Delta^m} \times A_i & \longrightarrow & Y_\alpha \\ \downarrow & & \downarrow \\ \underline{\Delta^m} \times A_i & \longrightarrow & * \end{array}$$

where I is a set of indices for representatives A_i of isomorphism classes of smooth schemes over S (therefore the vertical map on the left runs over the set of maps with respect to which we localize). We observe two things: the map between the coproducts is a cofibration in $\mathcal{S}$ and it induces isomorphisms in the first n simplicial dimensions. The fact that $\underline{\partial\Delta^m} \times A_i \to \underline{\Delta^m} \times A_i$ is a cofibration comes from the pushout-product axiom in the definition of a simplicial model category, applied to the cofibrations $\underline{\partial\Delta^m} \to \underline{\Delta^m}$ and $\emptyset \to A_i$ (Remark 2.36). In particular, this homotopy pushout is a strict pushout (Proposition 3.2), and the map $Y_\alpha \to Y_{\alpha+1}$ is a cofibration inducing isomorphism in the first n simplicial dimensions. If $\lambda < \kappa$ is a limit ordinal and all Y_α for $\alpha < \lambda$ have already been constructed, Y_λ is defined as $\mathsf{hocolim}_{\alpha<\lambda} Y_\alpha$. This homotopy colimits is a strict colimit (tower of cofibrations) of maps inducing isomorphisms in the first n simplicial levels, therefore $Y \to Y_\lambda$ also has this property. Finally, we apply the same argument to $RY := \mathsf{hocolim}_{\alpha<\kappa} Y_\alpha$ to conclude that $Y \to RY$ induces isomorphisms in the n first simplicial levels.

Step 4. The last step is to show that $\mathsf{cosk}_n U_\bullet \to V$ is a weak equivalence in $\check{\mathcal{S}}_n$. More precisely, we will show it is already a weak equivalence in $\check{\mathcal{S}}$, by induction on n. If $n = 0$ then we just have a Čech complex (Lemma 2.18), which by definition is a weak equivalence in $\check{\mathcal{S}}$. Assume the claim is proved for all integers at most n. We must show that $\mathsf{cosk}_{n+1} U_\bullet \to V$ is a weak equivalence in $\check{\mathcal{S}}$. Writing it as the composition $\mathsf{cosk}_{n+1} U_\bullet \to \mathsf{cosk}_n U_\bullet \to V$, by hypothesis of induction it suffices to show that $\mathsf{cosk}_{n+1} U_\bullet \to \mathsf{cosk}_n U_\bullet$ is a weak equivalence in $\check{\mathcal{S}}$. We will prove at the end the following claim: this is a covering map in every simplicial level. Assuming that it holds, consider the bisimplicial presheaf formed in simplicial level k by the cosimplicial presheaf which is the Čech complex of the covering map $(\mathsf{cosk}_{n+1} U_\bullet)_k \to (\mathsf{cosk}_n U_\bullet)_k$. Its homotopy colimit in $\check{\mathcal{S}}$ computed first along cosimplicial indices and then along simplicial ones yields exactly V. Indeed:

- the homotopy colimit of the k-th row is the constant simplicial presheaf $(\mathsf{cosk}_n U_\bullet)_k$. Indeed, by construction of $\check{\mathcal{S}}$, since $(\mathsf{cosk}_{n+1} U_\bullet)_k \to (\mathsf{cosk}_n U_\bullet)_k$ is a covering map, the map induced from its Čech complex to the constant simplicial presheaf $(\mathsf{cosk}_n U_\bullet)_k$ is a weak equivalence.
- the homotopy colimit of the simplicial diagram (indexed by k) of constant simplicial presheaves $(\mathsf{cosk}_n U_\bullet)_k$ has homotopy colimit $\mathsf{cosk}_n U_\bullet$ in $\mathcal{S}$ as we have seen in the proof of Lemma 2.32 (since $\mathsf{cosk}_n U_\bullet$ is cofibrant by the description of Remark 2.36). And $\mathsf{cosk}_n U_\bullet \to V$ is a weak equivalence in $\check{\mathcal{S}}$ by hypothesis of induction.

This homotopy colimit is also equivalent to the diagonal D of the bisimplicial object. We have a weak equivalence $D \to V$, and we want to show that $\mathsf{cosk}_{n+1} U_\bullet \to V$ is a retract of this map, which will allow us to conclude that it is a weak equivalence in $\check{\mathcal{S}}$. We have the following diagram (the row in blue describes the map on the k-simplices, namely the diagonal map followed by a projection):

$$(\mathsf{cosk}_{n+1} U_\bullet)_k \xrightarrow{\Delta} (\mathsf{cosk}_{n+1} U_\bullet)_k \times_{(\mathsf{cosk}_n U_\bullet)_k} \cdots \times_{(\mathsf{cosk}_n U_\bullet)_k} (\mathsf{cosk}_{n+1} U_\bullet)_k \xrightarrow{\pi_1} (\mathsf{cosk}_{n+1} U_\bullet)_k$$

$$\begin{array}{ccccc} \mathsf{cosk}_{n+1} U_\bullet & \longrightarrow & D & \longrightarrow & \mathsf{cosk}_{n+1} U_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ V & = & V & = & V \end{array}$$

and the composition of the two maps in the top row is clearly the identity.

The last bit is to prove the claim that $\mathsf{cosk}_{n+1} U_\bullet \to \mathsf{cosk}_n U_\bullet$ is a covering map in every simplicial level k. The argument takes place in category of presheaves *over V*. For $k \leq n$, the map in question is just the identity, and $k = n+1$ we obtain the map from U_{n+1} to the $(n+1)$-th matching object, which is a covering map by definition of a hypercover. For $k > n$, we rewrite the map induced in level k as ${}_S\mathsf{Set}(\Delta^k, \mathsf{cosk}_{n+1} U_\bullet(-)) \to {}_S\mathsf{Set}(\Delta^k, \mathsf{cosk}_n U_\bullet(-))$; by adjunction this is the same as ${}_S\mathsf{Set}(\mathsf{sk}_{n+1}\Delta^k, U_\bullet(-)) \to {}_S\mathsf{Set}(\mathsf{sk}_n\Delta^k, U_\bullet(-))$. We have a pullback square

$$\begin{array}{ccc} {}_S\mathsf{Set}(\mathsf{sk}_{n+1}\Delta^k, U_\bullet(-)) & \longrightarrow & \prod {}_S\mathsf{Set}(\Delta^{n+1}, U_\bullet(-)) \\ \downarrow & & \downarrow \\ {}_S\mathsf{Set}(\mathsf{sk}_n\Delta^k, U_\bullet(-)) & \longrightarrow & \prod {}_S\mathsf{Set}(\partial\Delta^{n+1}, U_\bullet(-)) \end{array}$$

coming from a construction of $\mathsf{sk}_{n+1}\Delta^k$ by attaching $(n+1)$-dimensional simplicial cells to $\mathsf{sk}_n\Delta^k$ (so that it is expressed as a pushout). For more details on this pushout square, see Lemmata 4.7 and 4.11 in [DHI04].

The map on the right-hand side is a covering map by the case $k = n + 1$ seen above (the product of coverings maps is still a covering map, by the refinement axiom in a Grothendieck topology). Since base change preserves covering maps (by the pasting law for pullbacks and the definition of a covering map), this finishes the proof. □

Proposition A.6. *A Nisnevich sheaf of sets, viewed as an object of* Spc*, is Nisnevich local. In particular, the simplicial presheaf represented by a smooth scheme* $X \in \mathsf{Sm}_S$ *is Nisnevich local.*

Proof. The second part of the statement finishes the proof of Proposition 2.35, and follows directly from the first part of the statement and the fact that Nisnevich topology is subcanonical (stated as Proposition 2.13). To prove this first part, let X a Nisnevich sheaf of sets. Clearly, X is objectwise fibrant because constant simplicial sets are Kan complexes. This also implies that objectwise, X has no homotopy above dimension 0. Therefore, by Theorem A.5, X has Nisnevich descent if and only if it has Cech descent. The homotopy limit appearing in the definition of descent is just a strict limit, because here it involves only constant simplicial sets. So we have to check that for any Čech hypercover $\check{C}(\mathcal{U}) \to V$, we have a bijection $X(V) \to \lim_n X(\check{C}(\mathcal{U})_n)$. This holds because X is a sheaf. Indeed, the limit of the equalizer diagram in the definition of a sheaf is the same as the limit over the full Čech complex (one way to see this is given towards the middle of the proof of Theorem A.8). □

Remark A.7 (Another model for motivic Postnikov towers)*.* The model structures from Step 1 of the proof of Theorem A.5 can be used to provide an alternative construction of the motivic Postnikov towers of Theorem 5.7. Indeed, we can perform the same Bousfield localization with respect to maps $\underline{\partial\Delta^k} \times U \to \underline{\Delta^k} \times U$, but in the model category $\mathsf{Spc}_{\mathbb{A}^1}$ instead: let $\mathsf{Spc}_{\mathbb{A}^1,n}$ be this localization. Then, similarly to we have seen, fibrant objects in $\mathsf{Spc}_{\mathbb{A}^1,n}$ are exactly $\mathbb{A}^1$-local presheaves whose objectwise simplicial homotopy is trivial above dimension $n-1$. Moreover, the small object argument we used to show that we can choose our fibrant replacements so that they induce isomorphisms on the first n simplicial levels can be generalized to the factorization of any map as an acyclic cofibration followed by a fibration. Given an $\mathbb{A}^1$-connected $\mathbb{A}^1$-local object $Y \in \mathsf{Spc}_{\mathbb{A}^1}$, we build $Y[1]$ by factoring the map $Y \to *$ in the model category $\mathsf{Spc}_{\mathbb{A}^1,2}$ into an acyclic fibration $Y \xrightarrow{\sim} Y[1]$ that induces isomorphisms on the first two simplicial levels, followed by a fibration $Y[1] \to *$. Inductively, to construct $Y[n+1]$, repeat the same process by factoring the map $Y \to Y[n]$ in $\mathsf{Spc}_{\mathbb{A}^1,n+2}$. For all $n \in \mathbb{N}$, since fibrations in $\mathsf{Spc}_{\mathbb{A}^1,n}$ are also fibrations in $\mathsf{Spc}_{\mathbb{A}^1}$ (we have a left Bousfield localization), this produces a tower of fibrations and fibrant objects in $\mathsf{Spc}_{\mathbb{A}^1}$. In particular, their $\mathbb{A}^1$-homotopy sheaves are just the simplicial ones. The fibrancy condition in $\mathsf{Spc}_{\mathbb{A}^1,n+2}$, and the fact that the first $n+2$ simplicial levels of X and $Y[n+1]$ agree then gives us exactly the condition on $\mathbb{A}^1$-homotopy sheaves we need. For convergence, as in the proof of Theorem 5.7, the homotopy limit is computed as a strict limit, and therefore objectwise. Then, the fact that each map $Y \to Y[n]$ induces isomorphisms in the first $n+1$ simplicial levels allows us to conclude the proof as in Theorem 5.7.

A.3 Proof of the Nisnevich descent theorem

The goal of this subsection is to prove Theorem 2.34. For the proof of the hard implication, we will follow very closely the exposition of Dugger in [Dug01c], but we will admit one result in algebraic geometry, which is proved in details in the same paper. The theorem itself is due Morel and Voevodsky, and to Brown and Gersten.

Theorem A.8 (Nisnevich descent theorem)**.** *Assume S is Noetherian and of finite Krull dimension. Let* $Y \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ *be a non-empty presheaf of Kan complexes. Then* Y *satisfies Nisnevich hyperdescent if and only if* $Y(\emptyset) \simeq *$ *and for every elementary Nisnevich square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow p \\ U & \xrightarrow[\iota]{} & X \end{array}$$

the natural map

$$Y(X) \longrightarrow Y(U) \times^h_{Y(U \times_X V)} Y(V)$$

to the homotopy pullback is a weak equivalence of simplicial sets.

Proof. Here is a proof relying on several statements we prove right below.

Assume first that Y has Nisnevich hyperdescent. Then, given a Nisnevich square as in the statement, $\mathcal{W} := \{U \to X, V \to X\}$ is a Nisnevich cover of X, and therefore Y has descent with respect to the Čech hypercover $\check{C}(\mathcal{W}) \to X$, namely $\mathsf{map}(X, Y) \to \mathsf{map}(\check{C}(\mathcal{W}), Y)$ is a weak equivalence of simplicial sets (see the proof of Proposition 2.32 for the equivalence between descent and this condition). We have to show that $\mathsf{map}(X, Y) \to \mathsf{map}(U, Y) \times^h_{\mathsf{map}(U\times_X V, Y)} \mathsf{map}(V, Y) \simeq \mathsf{map}(U \amalg^h_{U\times_X V} V, Y)$ is a weak equivalence. By 2-out-of-3, it suffices to show that $\mathsf{map}(U \amalg^h_{U\times_X V} V, Y) \to \mathsf{map}(\check{C}(\mathcal{W}), Y)$ is a weak equivalence. This holds because map preserves (objectwise) weak equivalences in the first variable (Theorem 17.6.3 in [Hir03]), and $\check{\mathcal{W}} \to U \amalg^h_{U\times_X V} V \simeq U \amalg_{U\times_X V} V$ is an objectwise weak equivalence. Indeed, since $\check{\mathcal{W}} = \mathsf{cosk}_0(\check{\mathcal{W}})$ (by Lemma 2.18), objectwise the Cech complex has no homotopy above dimension 0, and from this description we also see that it is fibrant (Proposition 5.6, item (i)). In particular, $\check{\mathcal{W}} \to \pi_0(\check{\mathcal{W}}(-))$ is an objectwise weak equivalence (where the connected components are viewed as a constant simplicial presheaf). But these connected components are exactly $U \amalg_{U\times_X V} V$. Indeed, on $Z \in \mathsf{Sm}_S$ they are given by the coequalizer

$$((U \times_X U) \amalg (U \times_X V) \amalg (V \times_X U) \amalg (V \times_X V))(Z) \rightrightarrows U(Z) \amalg V(Z).$$

But since we have a Nisnevich square, the first term in the coequalizer can be replaced with just $(U \times_X V)(Z)$, just as in the proof of Proposition A.4. Be careful that the coproducts appearing here are taken in the category of presheaves and are not disjoint unions of schemes!

The only thing that remains to be verified is that $Y(\varnothing) \simeq *$. If Y was a sheaf, this would follow as in the proof of Proposition A.4. So we will reduce to this case. Sheafification is clearly a stalkwise weak equivalence. Using the same construction for sheaves as we have done for presheaves when we defined Spc (one can also use the Joyal model structure, namely Definition 6.1 for sheaves), we can consider a Nisnevich fibrant replacement of sheaves $Y^+ \to R(Y^+)$. Looking at the category of sheaves as a subcategory of the category of presheaves, we have obtained a stalkwise weak equivalence $Y \to R(Y^+)$ of Nisnevich local presheaves in Spc. Indeed, $R(Y^+)$ is still Nisnevich local when viewed as a presheaf: the simplicial mapping spaces are the same in both categories because sheaves form a full subcategory of the category of presheaves, and moreover Nisnevich locality is verified against hypercovers, which are morphisms of sheaves. Then it must be a Nisnevich local weak equivalence (Theorem 6.2) and even an objectwise weak equivalence since both the domain and codomain are local objects. It follows that $Y(\varnothing) \simeq R(Y^+)(\varnothing) = *$, since $R(Y^+)$ is a (strict) sheaf by construction.

For the other implication, consider a fibrant replacement of Y in Spc, namely $Y \overset{\sim}{\hookrightarrow} RY \twoheadrightarrow *$. If we show that the first map is actually an objectwise weak equivalence, we are done, because then for any hypercover $U_\bullet \to V$, the map $Y(V) \simeq RY(V) \to \mathsf{holim}_n\, RY(U_n) \simeq \mathsf{holim}_n\, Y(U_n)$ is a weak equivalence of simplicial sets, because RY satisfies Nisnevich descent by construction. Thus it suffices to show that for presheaves of Kan complexes A and B both having Nisnevich excision, any Nisnevich weak equivalence $A \to B$ is an objectwise weak equivalence. But by usual homotopy theory, $A \to B$ is a Nisnevich (respectively objectwise) weak equivalence if and only if all homotopy fibers are Nisnevich (respectively objectwise) weakly equivalent to the point. Moreover the homotopy fibers also have Nisnevich excision, since the homotopy limits defining them commute with the homotopy pullbacks in the excision property. So it suffices to prove that for any simplicial presheaf C with Nisnevich excision, if $C \to *$ is a Nisnevich weak equivalence then it is an objectwise weak equivalence. This is almost exactly the statement of Theorem A.13 below, except that the latter applies to stalkwise weak equivalences (see Definition 6.1). However, every Nisnevich weak equivalence is a stalkwise weak equivalence, by Theorem 6.2. □

To prove Theorem A.13, which is the core of the proof of Theorem A.8, we will need to show that the Zariski topology also has the Brown-Gersten property, i.e. that we have the following result (by a similar reformulation as in the Nisnevich case):

Theorem A.9 (Zariski descent theorem). *Let $X \in {}_S\mathsf{Pre}(\mathsf{Sm}_S)$ be a presheaf that satisfies the Zariski excision property, namely the Nisnevich excision property for Zariski covers with two elements. If $X \to *$ is a (Zariski) stalkwise weak equivalence, then $X \to *$ is an objectwise weak equivalence.*

To talk about stalks in the Nisnevich topology it is also useful to have a notion of neighborhood:

Definition A.10 (Nisnevich neighborhood)**.** For $U \in Sm_S$ and $p \in U$ a point, a *Nisnevich neighborhood* (V, q) of p in U is a pair consisting of an étale map $f : V \to U$, and a point $q \in V$ such that $f(q) = p$ and f induces an isomorphism of residue fields at these points.

Following [Dug01c], we define:

Definition A.11 ((Strong) refinement conditions)**.** For any $X \in {}_S\mathrm{Pre}(Sm_S)$, we define the following properties for all $n \geq -1$:

- (R_n): Given any $U \in Sm_S$, and $p \in U$ of codimension n (i.e. $\mathcal{O}_{U,p}$ has Krull dimension n), for any $k \in \mathbb{N}$ and simplicial map $s : \partial\Delta^k \to X(U)$, there exists a Zariski open neighborhood V of p in U such that the restriction of s to V (namely the composition $\partial\Delta^k \to X(U) \to X(V)$) extends to Δ^k (we say that s extends over V).
- (R_n^{Nis}): Given any $U \in Sm_S$, and $p \in U$ of codimension n, for any $k \in \mathbb{N}$ and simplicial map $s : \partial\Delta^k \to X(U)$, there exists a Nisnevich neighborhood (V, q) of p such that the restriction of s to V extends to Δ^k.
- (SR_n): Condition (R_n) holds for p replaced with any set of points of codimension at most n in U.

(where R stands for refinement and SR for strong refinement).

These conditions clearly encode some kind of contractibility of stalks, since a fibrant simplicial set K has no homotopy if and only if every map $\partial\Delta^k \to K$ extends to Δ^k, but in the conditions this lift is only permitted in a smaller neighborhood. We will make this precise in Lemma A.26.

Proof of Theorem A.9. We may assume that X is objectwise fibrant, because taking an objectwise fibrant replacement yields a presheaf which is objectwise weakly equivalent to X and has the same properties (the homotopy limit in the definition of the excision property is preserved by objectwise weak equivalences).

Step 1. By Lemma A.26, since X is stalkwise contractible, it has property (R_n) for all $n \geq -1$.

Step 2. We will show that X has property (SR_n) for all $n \geq -1$ by induction on n. First of all X has property (SR_{-1}). Indeed, by definition of property (SR_{-1}), the set of points under consideration will necessarily be empty, and then we may choose $V = \emptyset$. The condition then holds because by the Zariski excision property $X(\emptyset)$ is contractible (and we have assume it was fibrant), so any map from the boundary of the standard k-simplex extends to the full k-simplex.

Then, we prove that X having Zariski excision and properties (SR_{n-1}) and (R_n) together, for some fixed $n \geq 0$, implies that X has property (SR_n). We will need this fact in the proof of Theorem A.8, and thus we do not allow ourselves to use the fact that X has properties (R_k) with $k > n$. By definition (SR_{n-1}) implies refinement properties $(R_0), \ldots, (R_{n-1})$ (and also the strong refinement properties). Consider $U \in Sm_S$ with a set T of points of codimension at most n, and $s : \partial\Delta^k \to X(U)$. If $S = \emptyset$, pick an irreducible component of X of maximal dimension. This is possible because U is Noetherian (so it has finitely many irreducible components), because S is Noetherian and U is smooth of finite type over S. In this setting, S being finite dimensional implies that U has the same property. The generic point η of this irreducible component has codimension 0 in U, and thus by (R_0) there exists a open neighborhood V of η in U such that $s : \partial\Delta^k \to X(V)$ extends to Δ^k, as desired. Now if $S \neq \emptyset$, by Noetherianity we may pick $V \subseteq U$ a Zariski open subset maximal with the property that $V \cap S \neq \emptyset$ and $s : \partial\Delta^k \to X(V)$ extends to Δ^k. A non empty open set satisfying this condition exists, because we can apply property $(R_{\mathrm{codim}_U(p)})$ to some point $p \in S$ (which has codimension less than n by assumption). This suffices to conclude because maximality forces $S \subseteq V$: indeed if there exists $q \in S \setminus U$, by $(R_{\mathrm{codim}_U(q)})$ and Lemma A.12 there is $W \supsetneq V$ over which the restriction of s extends, contradicting maximality.

Step 3. We conclude from there. Let $U \in Sm_S$. We have to show that $X(U) \to *$ is a weak equivalence of simplicial sets, namely that any map $\partial\Delta^k \to X(U)$ can be extended to Δ^k, for all $k \geq 0$. Since X has properties (R_n) for all $n \geq 0$, just as in step 2 above, there exists a non-empty open set V in U such that $s : \partial\Delta^k \to X(V)$ extends to Δ^k. Since U is moreover Noetherian, we can choose V maximal with this property. For a contradiction, assume that $V \neq U$. Then there is a point $p \in U \setminus V$ with finite codimension m. By property (R_m), there is a neighborhood W of p in U such that s extends over W as well. Lemma A.12 directly provides a contradiction. Thus $V = U$, and s extends to $\Delta^k \to X(U)$, as desired.

□

Lemma A.12. *Let $X \in {}_S\mathrm{Pre}(\mathrm{Sm}_S)$ be a presheaf of Kan complexes, $U \in \mathrm{Sm}_S$, and $s : \partial\Delta^k \to X(U)$. Assume that there is a Zariski open set $V \subseteq U$ such that $s : \partial\Delta^k \to X(V)$ extends to Δ^k, and that s also extends over some open neighborhood of some point $p \in U \setminus V$. If X has Zariski excision and property $(SR_{\mathrm{codim}(p)-1})$ then there exists $V' \subseteq U$ Zariski open, with $\{p\} \cup V \subseteq V'$, such that s extends over V'.*

Proof. This is Lemma 1.2 in [Dug01c]. We will not reproduce the proof, because we will show later a similar statement for the Nisnevich topology, and the argument follows the same pattern. □

We can finally prove:

Theorem A.13 (Reformulation of the Nisnevich descent theorem)**.** *Let $X \in {}_S\mathrm{Pre}(\mathrm{Sm}_S)$ be a presheaf that satisfies the Nisnevich excision property. If $X \to *$ is a stalkwise weak equivalence, then $X \to *$ is an objectwise weak equivalence.*

Proof. As before we may assume that X is a presheaf of Kan complexes by taking an objectwise fibrant replacement. Using the (proof of the) Zariski descent theorem A.9, it suffices to show that X has property (R_n) for all $n \geq -1$ (actually, by Lemma A.26, this implies that $X \to *$ is a stalkwise weak equivalence with respect to the Zariski topology). Indeed, X has Zariski excision because Zariski covers with two elements form in particular Nisnevich squares.

Step 1. Since X is Nisnevich stalkwise contractible, by Lemma A.26 it has all properties (R_n^{Nis}).

Step 2. We prove that for any fixed $n \geq -1$, properties (SR_n) and (R_{n+1}^{Nis}) together with Nisnevich excision imply property (R_{n+1}) (for objectwise fibrant presheaves). Let $U \in \mathrm{Sm}_S$, and $p \in U$ of codimension $n+1$, and consider $s : \partial\Delta^k \to X(U)$. By property (R_{n+1}^{Nis}), there is a Nisnevich neighborhood $(f : V \to U, q)$ of p in U such that s extends over V. Then, there are Zariski open neighborhoods V' of q in V and U' of p in U, with $f(V') \subseteq U'$ and f inducing an isomorphism $f^{-1}(\overline{\{p\}}) \to \overline{\{p\}}$. This is the statement in algebraic geometry that we said we would admit, see Proposition A.1 in [Dug01c] for a proof. Since we must only find *some* Zariski open neighborhood of p with the right properties, we may as well assume $U = U'$ and $V = V'$. The argument to come is quite heavy in notation; to keep track of it we include an illustration of the situation below. We now use property (SR_n) for the set T of points in $U \setminus \overline{\{p\}}$ such that p lies in their closure. We obtain a Zariski open set $W \subseteq U \setminus \overline{\{p\}}$ such that $W \supseteq T$ and s extends on W. By Lemma A.14 below, there is a Zariski open neighborhood W' of p in U such that $W' \cap U \setminus \overline{\{p\}} = W$. We claim that we have created a Nisnevich square $\{W \to W', V \times_U W' \to W'\}$. Indeed, the second map is étale as a base change of f, which is étale. By construction W is Zariski open in W', and moreover π is a isomorphism over $W' \setminus W = \overline{\{p\}}$: this (co)restriction is just $f : f^{-1}(\overline{\{p\}}) \to \overline{\{p\}}$, which is an isomorphism by hypothesis. Moreover s extends over W and V by construction. By Nisnevich excision, we obtain a weak equivalence $X(W') \to X(W) \times^h_{X(V \times_U W' \times_{W'} W)} X(V \times_U W') = X(W) \times^h_{X(V \times_U W)} X(V \times_U W')$. This yields a long exact sequence:

$$\cdots \to \pi_{k+1} X(V \times_U W) \to \pi_k X(W') \to \pi_k X(W) \times \pi_k X(V \times_U W') \to \ldots$$

(indeed, a square is a homotopy pullback if and only if the homotopy fiber of the horizontal morphisms are weakly equivalent; one can write down the two long exact sequences for the corresponding fiber sequences, and compare them via the isomorphism induced between the homotopy fibers; the result then follows from homotopical algebra).

Then, the homotopy class of s lies in the image of $\pi_{k+1} X(V \times_U W)$ (note that $V \times_U W = f^{-1}(W)$), because its restriction to W and $V \times_U W'$ is trivial (since $V \times_U W' \to U$ factors through V, over which s extends). So we may pick a lift $t : \partial\Delta^{k+1} \to X(V \times_U W)$ for s. We apply property (SR_n) to the points in $V \times_U W$ that lie in the closure of q in V. Since q has codimension $n+1$ (because p has codimension $n+1$, and étale maps are smooth of relative dimension 0), all points not equal to q lying in its closure have codimension at most n. So there is $V' \subseteq V \times_U W$ Zariski open over which t extends. By Lemma A.14, q has a Zariski open neighborhood $V'' \subset f^{-1}(W')$ with $V'' \cap f^{-1}(W) = V'$.

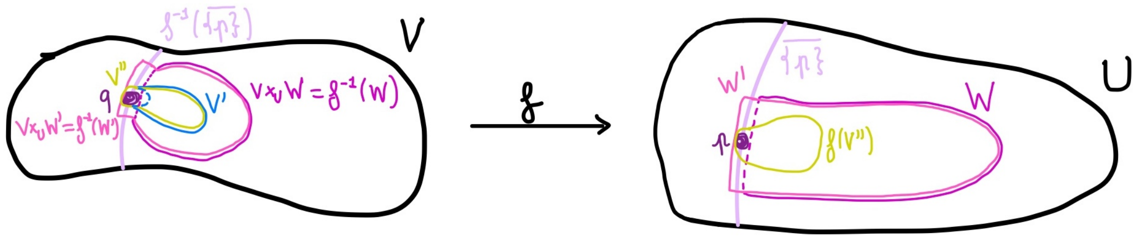


Figure 6: What we hope is a helpful illustration of the situation

We have another Nisnevich square formed by the maps $\{W \to W \cup f(V''), V'' \to W \cup f(V'')\}$ (it is just the base change to $W \cup f(V'')$ of the previous Nisnevich square). Therefore we have a weak equivalence $X(W \cup f(V'')) \to X(W) \times_{X(W \times_U V'')} X(V'')$, yielding as before a long exact sequence:

$$\cdots \to \pi_{k+1} X(W \times_U V'') \to \pi_k X(W \cup f(V'')) \to \pi_k X(W) \times \pi_k X(V'') \to \ldots$$

which we can compare to the previous one via the morphisms induced by the inclusions:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_{k+1} X(V \times_U W) & \xrightarrow{a} & \pi_k X(W') & \longrightarrow & \pi_k X(W) \times \pi_k X(V \times_U W') & \longrightarrow & \cdots \\ & & \downarrow b & & d \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_{k+1} X(W \times_U V'') & \xrightarrow{c} & \pi_k X(W \cup f(V'')) & \longrightarrow & \pi_k X(W) \times \pi_k X(V'') & \longrightarrow & \cdots \end{array}$$

We have $a([t]) = [s]$, so $d([s]) = c(b([t])) = 0$ since t extends over $W \times_U V'' = f^{-1}(W) \cap V'' = V'$ by construction. This shows that s extends over $W \cup f(V'') \ni p$, as desired ($f(V'')$ is open in U since f is étale and in particular open).

Step 3. We conclude by showing by induction that X has property (R_n) for all $n \geq -1$. The initialization step is trivial because (R_{-1}) is an empty condition (there are no points of codimension -1). As discussed in Step 2 of the proof of Theorem A.9, the fact that $X(\emptyset) \simeq *$ implies property (SR_{-1}). Our hypothesis of induction is that X satisfies (R_k) and (SR_k) for all $k \leq n$, for a fixed integer $n \geq -1$. Then by combining Step 1 and Step 2, X has property (R_{n+1}). Using Step 2 in the proof of Theorem A.9, since X is objectwise fibrant and has in particular Zariski excision, properties (SR_n) and (R_{n+1}) together implies property (SR_{n+1}), which finishes the induction.

□

Here is a "topological" lemma we have used:

Lemma A.14. *Let $W \subseteq U \subseteq Y$ be open inclusions of schemes in Sm_S, and $p \in Y \setminus U$. If every point $q \in U$ whose closure contains p lies in W, then p has a Zariski open neighborhood V in Y such that $W = U \cap V$.*

Proof. Let C be the closed complement of W in Y. Since S is Noetherian, we have seen that all schemes in Sm_S are Noetherian too, and therefore C is a finite union of its irreducible components; it can be written as the union of the closures of their generic points $\overline{\{\eta_1\}} \cup \cdots \cup \overline{\{\eta_n\}}$. Let C' be the union of all $\overline{\{\eta_i\}}$ such that $p \notin \overline{\{\eta_i\}}$, for $i \leq n$. We claim that $V := Y \setminus C'$ is the open neighborhood we are looking for. Indeed, it contains p, and $W \subseteq V \cap U$ because $W \setminus Y = W \cap C' \subseteq W \cap C = \emptyset$ by construction. To show the other inclusion, we check that $C \setminus C' \subseteq Y \setminus U$. If we know this, then $C \subseteq C' \cup (Y \setminus U)$, therefore $Y \setminus (C' \cup (Y \setminus U)) \subseteq Y \setminus C$, or equivalently $(Y \setminus C') \cap U \subseteq Y \setminus C$ and finally $V \cap U \subseteq W$. To prove the claim, let $x \in C' \setminus C$, and pick $j \leq n$ with both x and p contained in $\overline{\{\eta_j\}}$. Then $\eta_j \notin U$, else by assumption on W we have $\eta_j \in W$, whereas η_j belongs to C the complement of W. Now $\eta_j \notin U$ implies that U does not intersect $\overline{\{\eta_j\}}$, because any non empty open set has to contain the generic point. In particular, $x \in \overline{\{\eta_j\}}$ cannot be in U. □

A.4 Stalks of presheaves in the Zariski and Nisnevich topology

A.4.1 Generalities

In order to be able to work rigorously with stalks of presheaves over Sm_S, we have to introduce the notion of a point in a Grothendieck site. Example A.18 below justifies the definition.

Definition A.15 (Stalk functor)**.** Let $(\mathcal{C}, \tau)$ be a site, and let $p : \mathcal{C} \to \mathsf{Set}$ be a functor. The *stalk functor* $\bullet_p : \mathsf{Pre}(\mathcal{C}) \to \mathsf{Set}$ associated with p is defined on objects by

$$\mathcal{F} \mapsto \mathcal{F}_p := \mathsf{colim}_{(U,x)} \mathcal{F}(U),$$

where the colimit is indexed by the opposite category of the category whose objects are pairs (U, x) with $U \in \mathcal{C}$ and $x \in p(U)$, and a morphism between two objects (U, x) and (V, y) is a morphism $f : U \to V$ in $\mathcal{C}$ such that $p(f)(x) = y$. This definition can be extend to presheaves with values in any category with arbitrary colimits.

Definition A.16 (Point in a site)**.** A *point* in a site $(\mathcal{C}, \tau)$ is a functor $p : \mathcal{C} \to \mathsf{Set}$ such that:

- For any covering $\{U_i \to U\}_{i \in I} \in \tau$, the induced map $\coprod_{i \in I} p(U_i) \to p(U)$ is surjective.
- For any covering $\{U_i \to U\}_{i \in I} \in \tau$ and morphism $V \to U$ in $\mathcal{C}$, for all $i \in I$ the natural map $p(U_i \times_U V) \to p(U_i) \times_{p(U)} p(V)$ is a bijection.
- The associated *stalk functor* $\bullet_p$ is left exact, namely it preserves finite limits.

Remark A.17 (Points versus stalk functors). Every point defines a stalk functor $\mathsf{Shv}(\mathcal{C}) \to \mathsf{Set}$; actually the latter uniquely determines the point. This functor as specific properties, and can be viewed as a *point of the topos* $\mathsf{Shv}(\mathcal{C})$. With this point of view, the points of the topos of sheaves on a site correspond exactly to the points of the site; and thus points in a site can be described by the stalk functors. A reference for this language is [AGV72], IV.6.

Example A.18. Consider the site $\mathsf{Op}(X)$ of open subsets of some topological space X. A point in the site corresponds exactly to a point $x \in X$ in the usual sense, via the following correspondence. An element $x \in X$ defines a functor $\mathsf{Op}(X) \to \mathsf{Set}$ assigning the empty set to an open subset of X that does not contain x, and a one-element set otherwise.

Example A.19. In the notation above, the stalk of a representable presheaf $\mathcal{C}(-, c)$ at p is exactly $p(c)$.

In algebraic geometry, one is used to check isomorphisms between sheaves on a fixed scheme at their stalks. In the theory of more general sites, this is not always possible. The right notions are the following ones:

Definition A.20. A family of points $\{p_i\}_{i \in I}$ in a site $(\mathcal{C}, \tau)$ is called *conservative* if every map $\mathcal{F} \to \mathcal{F}'$ of sheaves on $\mathcal{C}$ such that $\mathcal{F}_{p_i} \to \mathcal{F}'_{p_i}$ is a bijection for all $i \in I$ is an isomorphism of sheaves.

A site $(\mathcal{C}, \tau)$ *has enough points* if it admits a conservative family of points.

Example A.21. The Zariski and Nisnevich topologies on Sm_S have enough points when S is Noetherian of finite Krull dimension. We will describe a conservative family of points for these sites in Proposition A.24.

We will use the following criterion:

Proposition A.22 (Criterion for a conservative family of points ([AGV72], IV.6.5))**.** *Let $\{p_i\}_{i \in I}$ be a family of points in a site $(\mathcal{C}, \tau)$. Assume that for any family $\{U_j \to U\}_{j \in J}$ in $\mathcal{C}$ such that $\{(U_j)_{p_i} \to U_{p_i}\}_{j \in J}$ is jointly surjective for all $i \in I$, the family $\{U_j \to U\}_{j \in J}$ is a covering with respect to τ. Then $\{p_i\}_{i \in I}$ is a conservative family of points. The converse statement is also true.*

Note that this criterion does not require us to know the actual points p_i, we only have to understand the associated stalk functors.

A.4.2 Points in the Zariski and Nisnevich sites of smooth schemes

For our purposes we only have to describe points in the Nisnevich and Zariski sites over S. We are insterested in the site of smooth schemes over S, but we begin with a proposition for the site of all schemes over S. In this situation, a general description is provided in [GK15]:

Proposition A.23 (Points for Sch_S)**.** *If S is a separated Noetherian scheme and τ is a topology on the category Sch_S of all schemes over S, then there is a bijection (which can be made into an equivalence of categories) between τ-local affine schemes (defined below) and points of the site Sch_S. This bijection sends a scheme $P \to S$ to a point of the site (Sch_S, τ) whose associated stalk functor is $\mathcal{F} \mapsto \mathcal{F}(P) \cong \mathsf{colim}_{P \downarrow \mathsf{Sch}_S} \mathcal{F}$.*

A scheme P over S is called τ-local if any covering $\{U_i \to U\}$ with respect to τ induces a surjection $\coprod_i \mathsf{Sch}_S(P, U_i) \to \mathsf{Sch}_S(P, U)$. Note the similarity with the criterion of Proposition A.22. This will be useful to us in the proof of Proposition A.24.

In [GK15], local schemes are described for a long list of topologies. In our case, we just need to know that Zariski-local affine schemes are the spectra of local rings, and Nisnevich-local affine schemes are the spectra of local Henselian rings. A local ring $(R, \mathfrak{m})$ is called Henselian if for every $f \in R[t]$ monic, and every root a_0 of the reduction of f modulo $\mathfrak{m}$ such that $f'(a_0) = 0 \mod \mathfrak{m}$, there exists a root $a \in R$ of f which reduces to a_0 modulo $\mathfrak{m}$. Namely, roots of a monic polynomial over the residue field can be lifted to roots of a monic lift of the polynomial over the ring itself.

We now specialize to the site of smooth schemes. Note that Sm_S in a subcategory of Sch_S, and the inclusion functor respects the Zariski and Nisnevich topologies, fiber products, and the terminal object. Therefore any point $p : \mathsf{Sch}_S \to \mathsf{Set}$ induces a point of Sm_S by post-composition with the inclusion functor. In terms of stalk functors, this inclusion induces a functor $F : \mathsf{Shv}(\mathsf{Sch}_S) \to \mathsf{Shv}(\mathsf{Sm}_S)$. Every stalk functor coming from a point of Sch_S therefore factors as the composition of F and the stalk functor corresponding to the induced point of Sm_S (this is about viewing points of a site as points in the topos of sheaves on this site, and about studying morphism of topoi). This relation between points of Sch_S and of Sm_S hopefully makes at least believable the fact that the functors in the following result actually define points:

Proposition A.24 (Points for Sm_S ([MV99] p 99))**.** *Let S be a Noetherian scheme of finite Krull dimension. The Zariski site $(\mathsf{Sm}_S, \tau_{\mathsf{Zar}})$ admits a conservative family of points $\{p_{(U,u)}\}$ indexed by all $U \in \mathsf{Sm}_S$ and points $u \in U$ in the scheme-theoretic sense, whose associated fiber functors are given by:*

$$\begin{aligned} \bullet_{p_{U,u}} =: \bullet_u : \mathsf{Pre}(\mathsf{Sm}_S) &\longrightarrow \mathsf{Set} \\ \mathcal{F} &\longmapsto \operatorname{colim}_{\mathsf{Spec}(\mathcal{O}_{U,u}) \to X \to S} \mathcal{F}(X) \end{aligned}$$

where the colimit is indexed by factorizations in Sch_S of the map $\mathsf{Spec}(\mathcal{O}_{U,u}) \to S$ such that $X \in \mathsf{Sm}_S$.

This colimit can equivalently be computed as $\operatorname{colim}_{u \in V \subseteq U} \mathcal{F}(V)$ where the colimit is indexed by the Zariski open neighborhoods V of u in U (in particular, this is just the usual stalk at u of $\mathcal{F}$ viewed as a presheaf on the scheme U). Moreover, if $Z \in \mathsf{Sm}_S$, we have $Z_u \cong \mathsf{Sch}_S(\mathsf{Spec}(\mathcal{O}_{U,u}), Z)$.

The Nisnevich site $(\mathsf{Sm}_S, \tau_{\mathsf{Nis}})$ admits a conservative family of points $\{p_{(U,u)}\}$ indexed by all $U \in \mathsf{Sm}_S$ and points $u \in U$ in the scheme-theoretic sense, whose associated fiber functors are given by:

$$\begin{aligned} \bullet_{p_{U,u}} =: \bullet_u : \mathsf{Pre}(\mathsf{Sm}_S) &\longrightarrow \mathsf{Set} \\ \mathcal{F} &\longmapsto \operatorname{colim}_{\mathsf{Spec}(\mathcal{O}^{\mathsf{hen}}_{U,u}) \to X \to S} \mathcal{F}(X) \end{aligned}$$

where the colimit is indexed by factorizations in Sch_S of the map $\mathsf{Spec}(\mathcal{O}^{\mathsf{hen}}_{U,u}) \to S$ such that $X \in \mathsf{Sm}_S$, where "hen" denotes Henselianization.

This colimit can equivalently be computed as $\operatorname{colim}_{(V \to U, v)} \mathcal{F}(V)$ where the colimit is indexed by the Nisnevich neighborhoods $(V \to U, v)$ of u in U see Definition A.10). Moreover, if $Z \in \mathsf{Sm}_S$, we have $Z_u \cong \mathsf{Sch}_S(\mathsf{Spec}(\mathcal{O}^{\mathsf{hen}}_{U,u}), Z)$.

Proof. Since Z viewed as a presheaf on Sm_S is just the restriction of the presheaf defined by Z on Sch_S (Zariski topology is subcanonical for the site of all schemes over S), if we admit that the point $p_{U,u}$ is induced by the point of Sch_S corresponding to $\mathsf{Spec}(\mathcal{O}_{U,u})$, then by the discussion above the stalk of Z at $p_{U,u}$ is equal to its stalk at the point corresponding to $\mathsf{Spec}(\mathcal{O}_{U,u})$, and we saw that this was $Z(\mathsf{Spec}(\mathcal{O}_{U,u})) = \mathsf{Sch}_S(\mathsf{Spec}(\mathcal{O}_{U,u}), Z)$. The same holds for the Nisnevich topology.

We will not prove that these functors indeed define points, but we will prove that they form a conservative family using the criterion of Proposition A.22. The argument below also show the equivalents formulas for the stalks, since it will show that the category indexing the colimits in the equivalent formulas is cofinal in the category indexing the colimits of the definition.

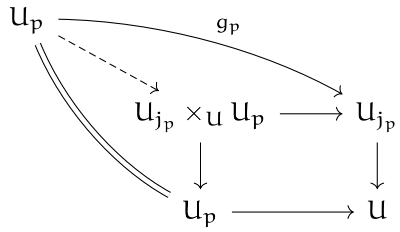
For the Zariski site, consider $\{f_j : U_j \to U\}_{j\in J}$ an arbitrary family of maps in Sm_S. Assume that for any $X \in \mathsf{Sm}_S$ and $x \in X$, the family $\{(U_j)_x \to U_x\}_{j\in J}$ is jointly surjective, i.e. it induces a surjection $\coprod_{j\in J} \mathsf{Sch}_S(\mathsf{Spec}(\mathcal{O}_{X,x}), U_j) \to \mathsf{Sch}_S(\mathsf{Spec}(\mathcal{O}_{X,x}), U)$. We have to show that $\{U_j \to U\}_{j\in J}$ is a covering in the Zariski topology. For all $p \in U$, the surjectivity of this map for the point corresponding to (U, p) implies that there exists $j_p \in J$ such that the natural map $\mathsf{Spec}(\mathcal{O}_{U,p}) \to U$ factors through U_{j_p}.

We claim that the map $g'_p : \mathsf{Spec}(\mathcal{O}_{U,p}) \to U_{j_p}$ we obtained factors through some Zariski open neighbourhood U_p of p in U. This will define a map $g_p : U_p \to U_{j_p}$. Since $\mathcal{O}_{U,p}$ is a local ring, and therefore is local in the sense of Proposition A.23, for any open cover of U_{j_p} by affine schemes, the map g'_p factors through one of these affine open subschemes. Therefore, by restricting to affine open subsets in S, U, and U_p, our problem translates into the following algebraic question:

Given a ring R, finitely presented R-algebras A and B, and a prime ideal $\mathfrak{p}$ in A, does any ring homomorphism $B \to A_\mathfrak{p}$ factor through A_f for some $f \in A$?

By hypothesis, B admits a finite presentation $R[x_1, \dots, x_n]/(f_1, \dots, f_m) \cong B$. The images of the generators x_i by the corresponding map $R[x_1, \dots, x_n] \twoheadrightarrow B \to A_\mathfrak{p}$ might have a non-trivial denominator in $A \setminus \mathfrak{p}$. Collect all these denominators and call $f \in A \setminus \mathfrak{p}$ their product. Then $R[x_1, \dots, x_n] \to A_\mathfrak{p}$ factors through A_f as a morphism of R-algebras. For all $i \leq m$, we can write $\phi(f_i) = a_i/(f^{n_i})$ for some $n_i \in \mathbb{N}$. Since the image of $\phi(f_i)$ in $A_\mathfrak{p}$ has to be zero by construction, there exists $r_i \in A \setminus \mathfrak{p}$ with $r_i a_i = 0$. Replacing f with $fr_1 \cdots r_m$, the morphism $R[x_1, \dots, x_n] \to A_f$ now passes to the quotient by $(f_1, \dots, f_m)$, which is just B. This answers the question.

We therefore have created a Zariski covering $\{U_p \hookrightarrow U\}_{p\in U}$ in Sm_S (if U is smooth over S, so is any Zariski open subset of U with the induced structure map). By the "refinement" axiom of a Grothendieck topology, to conclude that $\{f_j : U_j \to U\}_{j\in J}$ is a covering in the Zariski topology, we only have to show that $\{U_j \times_U U_p \to U_p\}_{j\in J}$ is a covering for all $p \in U$. This holds by the "section" axiom of a topology: indeed, the map $U_{j_p} \times_U U_p \to U_p$ admits a section, because we have by the universal property of the fiber product a morphism as follows:



The proof in the Nisnevich case is the same, if we can show that any map $\mathsf{Spec}(\mathcal{O}^{\mathsf{hen}}_{U,p}) \to U_{j_p}$ factors through some Nisnevich open neighbourhood (V_p, q_p) of p in U. Again this reduces to the following algebraic problem:

Given a ring R, finitely presented R-algebras A and B, and $\mathfrak{p}$ prime in A, does any ring homomorphism $B \to (A_\mathfrak{p})^{\mathsf{hen}}$ factor through some scheme S, with an étale ring homomorphism $A \to S$ and a prime $\mathfrak{q}$ in S lying over $\mathfrak{p}$, such that it induces an isomorphism at their residue fields?

The latter condition is just the algebraic notion of a Nisnevich neighborhood. By Lemma 10.155.7 in the Stacks project (Tag 0BSK), $(A_\mathfrak{p})^{\mathsf{hen}}$ is exactly the filtered colimit of such rings S over the pairs $(A \to S, \mathfrak{q})$. Thus it suffices to show the morphism from B factors through some stage of the colimit. The argument is similar to the one we used in the Zariski case: find lifts for the images of the finitely many generators of B over R, and again the ideal encoding the relations between them is finitely generated. The colimit being filtered, one can find a scheme S in which all generators admit lifts and the images of all generators of the kernel are trivial (take the tensor product over A of all the schemes in which lifts have been chosen). □

A.4.3 Contractibility of stalks

We begin by noting that the conservative families of points of Proposition A.24 are "conservative for simplicial presheaves" as well:

Remark A.25. If $\mathcal{F} \to \mathcal{F}'$ is a morphism of stalkwise (or objectwise) fibrant simplicial presheaves on Sm_S, such that the induced map $\mathcal{F}_{p_i} \to \mathcal{F}'_{p_i}$ is a weak equivalence of simplicial sets for all p_i in the conservative family of points for the Zariski, respectively Nisnevich topology described in Proposition A.24, then $\mathcal{F}_p \to \mathcal{F}'_p$ is a weak equivalence of simplicial sets for all points p of the site.

Indeed, what we want to show is that $\mathcal{F} \to \mathcal{F}'$ is a combinatorial weak equivalence in the sense of Jardine (see Definition 6.1). By Proposition 1.18 in [Jar87a], and one of the equivalent definition of a combinatorial weak equivalence given p 48 of the same reference, it suffices to show that there are induced isomorphisms of the sheaves associated with the presheaves $\pi_k(\mathcal{F}|_U(-))$ and $\pi_k(\mathcal{F}'|_U(-))$ for all $U \in \mathsf{Sm}_S$ and basepoints $x \in \mathcal{F}(U)_0$.

Pick $U \in \mathsf{Sm}_S$. By hypothesis, $\mathcal{F}_{p_i} \to \mathcal{F}'_{p_i}$ is a weak equivalence for all $i \in I$. Restricting ourselves to points $p_{(V,\nu)}$ of Sm_S such that $V \in \mathsf{Sm}_U$, we have weak equivalences $(\mathcal{F}|_U)_{p_j} \to (\mathcal{F}'|_U)_{p_j}$ for all points p_j of a conservative family of points in the Zariski, respectively Nisnevich site on Sm_U. Then there are isomorphisms of groups $\pi_k((\mathcal{F}|_U)_{p_j}) \cong \pi_k((\mathcal{F}'|_U)_{p_j})$ for any choice of basepoint for $(\mathcal{F}|_U)_{p_j}$ and the corresponding basepoint for $(\mathcal{F}'|_U)_{p_i}$. Since homotopy groups of simplicial sets commute with filtered colimits (and stalk functors are defined by such colimits), we have isomorphisms at stalks for all p_i of the presheaves of simplicial homotopy groups $\pi_k((\mathcal{F}|_U)(-))$ and $\pi_k((\mathcal{F}'|_U)(-))$ at any *global* basepoint (a vertex in $\mathcal{F}(U)$). Since sheafification preserves stalks, the same holds for the associated sheaves. Therefore the associated sheaves are isomorphic, because the family of points is conservative by assumption. This is what we needed.

Finally, we can show the Lemma we need to conclude the proof of Theorem A.8:

Lemma A.26 (Properties (R_n) and contractibility of stalks)**.** *Let X be simplicial presheaf of Kan complexes on Sm_S, with S Noetherian of finite Krull dimension. Then:*

(i) The presheaf X has property (R_n) for all $n \geq -1$ if and only if $X_p \simeq *$ for all points p in the site $(\mathsf{Sm}_S, \tau_{\mathsf{Zar}})$ (i.e. all its Zariski stalks are contractible).

(ii) The presheaf X has property (R_n^{Nis}) for all $n \geq -1$ if and only if $X_p \simeq *$ for all points p in the site $(\mathsf{Sm}_S, \tau_{\mathsf{Nis}})$ (i.e. all its Nisnevich stalks are contractible).

Proof. By Remark A.25, the condition about the contractibility of the stalks can in both cases be replaced by contractibility only at the stalks for the family of points from Proposition A.24.

For part (i), assume that X has property (R_n) for all $n \geq -1$. Pick a point as in Proposition A.24; it is indexed by some $U \in \mathsf{Sm}_S$ and $u \in U$. Since X_u is a Kan complex, it is contractible if and only if every map $\partial\Delta^k \to X_u$ extends to a map $\Delta^k \to X_u$. Pick a map $s : \partial\Delta^k \to X_u = \mathrm{colim}_{u \in V \subseteq U} \mathcal{X}(V)$. Lifting the image by s of each simplex in $\partial\Delta^k$ to $X(V)$ for some Zariski open neighbourhood V of u in U, and intersecting these neighborhoods, we obtain a map $\partial\Delta^k \to X(V')$ for some Zariski open neighborhood V' of u in U. Since all properties (R_n) hold (and u has finite codimension in V' as we have seen before), there exists a smaller Zariski open neighborhood $V'' \ni u$ such that the s extends over V''. The composition $s : \Delta^k \to X(V'') \to X_u$ provides the desired extension for s.

Conversely, if every stalk of X is contractible, for $U \in \mathsf{Sm}_S$ and a point $u \in U$, and any map $s : \partial\Delta^k \to X(U)$, the composition $s : \partial\Delta^k \to X(U) \to X_u = \mathrm{colim}_{u \in V \subseteq U} \mathcal{X}(V)$ extends to Δ^k by contractibility. Lifting the image of the unique non-degenerate k-simplex to some $X(V)$ for V a Zariski open neighborhood of p in U, we obtain an extension $s : \Delta^k \to X(V)$ as desired.

For part (ii), the proof is the same, except we replace the intersection of Zariski open neighborhoods in U by the fiber product over U of the Nisnevich neighborhoods obtained. $\square$

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