THE REAL BETTI REALIZATION OF MOTIVIC THOM SPECTRA AND OF VERY EFFECTIVE HERMITIAN K-THEORY

JULIE BANNWART

Abstract. Real Betti realization is a symmetric monoidal functor from the category of motivic spectra to that of topological spectra, extending the functor that associates to a scheme over $\mathbb{R}$ the space of its real points. In this article, we prove some results about the real Betti realizations of certain motivic $\mathcal{E}_1$- and $\mathcal{E}_\infty$-rings. We show that the motivic Thom spectrum functor and the topological one correspond to each other, as symmetric monoidal functors, under real realization. In particular, we obtain equivalences of $\mathcal{E}_\infty$-rings between the real realizations of the variants MGL, MSL, and MSp of algebraic cobordism, and the variants MO, MSO, and MU of topological cobordism, respectively. Using this, we identify the $\mathcal{E}_1$-ring structure on the real realization of ko, the very effective cover of Hermitian K-theory, by an explicit 2-local fracture square, as being equivalent to $\mathsf{L}(\mathbb{R})_{\geq 0}$, the connective L-theory spectrum of $\mathbb{R}$.

Contents

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Introduction

Motivic homotopy theory, mainly introduced by Morel and Voevodsky [MV99], provides a setting to study algebraic geometry and cohomology theories for schemes using the methods of homotopy theory. We work in the categories of motivic spaces $\mathsf{Spc}(S)$ and of motivic spectra $\mathsf{SH}(S)$ over a base scheme S. The former is obtained by considering the subcategory of presheaves of spaces on Sm_S (smooth schemes of finite type over S) consisting of those presheaves that are $\mathbb{A}^1$-invariant and are sheaves in the Nisnevich topology. Namely, we impose Nisnevich descent and contractibility of the affine line via a Bousfield localization. In a formula, $\mathsf{Spc}(S) = \mathsf{L}_{\mathbb{A}^1,\mathsf{Nis}}(\mathcal{P}(\mathsf{Sm}_S))$. The category of motivic spectra $\mathsf{SH}(S) = \mathsf{Spc}(S)_*[(\mathbb{P}^1)^{-1}]$ is then obtained from that of motivic spaces by $\mathbb{P}^1$-stabilization, i.e., by making the projective line invertible with respect to the smash product in the category of pointed motivic spaces. We obtain a presentably symmetric monoidal, stable category.

Several classical examples of cohomology theories for schemes become representable in the category of motivic spectra: for instance, motivic cohomology with integral coefficients, represented by $\mathsf{H}\mathbb{Z} \in \mathsf{SH}(S)$, motivic cohomology with $\mathbb{Z}/2$-coefficients $\mathsf{H}\mathbb{Z}/2$, ($\mathbb{A}^1$-invariant) algebraic K-theory KGL, ($\mathbb{A}^1$-invariant) Hermitian K-theory KO, algebraic cobordism MGL, and so on. All these examples are not only motivic spectra, but they are also endowed with multiplicative structures, which the corresponding cohomology theories inherit. More precisely, they all admit the structure of commutative algebra objects in $\mathsf{SH}(S)$, i.e., they can be made into motivic $\mathcal{E}_\infty$-rings.

To investigate motivic phenomena, it is sometimes useful to bring them back into the topological setting. The real Betti realization functor $r_\mathbb{R}$ is one of the tools to do so. This functor from the category of motivic spaces over $\mathbb{R}$ (or any subfield $k \hookrightarrow \mathbb{R}$) to that of spaces extends the functor of real points (from the category of smooth schemes over $\mathbb{R}$ to that of spaces). It stabilizes to a functor between the category of motivic spectra and that of topological spectra. Similarly, one can define a complex Betti realization functor $r_\mathbb{C}$ from motivic spaces over $\mathbb{C}$ (or any subfield) to spaces, which has a stable analog from motivic spectra over $\mathbb{C}$ to topological spectra. For example, the complex realization of KGL is equivalent to the topological spectrum KU representing complex topological K-theory [ARØ20, Lem. 2.3], while its real realization vanishes [BH20, Lem. 3.9]. The motivic spectrum KO complex realizes to $\mathsf{KO}^{\mathsf{top}}$ [ARØ20, Lem. 2.13], representing real topological K-theory, but its real realization is $\mathsf{KO}^{\mathsf{top}}[1/2]$ [BH20, Lem. 3.9]. The motivic spectra MGL, MSL, and MSp, representing variants of algebraic cobordism, have real realizations the spectra MO, MSO, and MU respectively, representing variants of topological cobordism [BH20, Cor. 4.7].

The latter examples of topological spectra also admit the structure of commutative algebra objects, namely they are topological $\mathcal{E}_\infty$-rings. Since the Betti realization functors are symmetric monoidal with respect to the smash products on motivic and topological spectra, the realization of any motivic $\mathcal{E}_\infty$-ring inherits a structure of topological $\mathcal{E}_\infty$-ring. It is therefore natural to wonder whether one can generalize the aforementioned result to the level of $\mathcal{E}_\infty$-rings. For example, is the real realization of MGL equivalent *as an $\mathcal{E}_\infty$-ring* to MO with its usual $\mathcal{E}_\infty$-structure? We will answer this question for some examples of classical motivic spectra.

Theorem A (Theorem 2.45)**.** *There are equivalences of $\mathcal{E}_\infty$-rings*

$$r_\mathbb{R}\mathsf{MGL} \simeq \mathsf{MO}, \qquad r_\mathbb{R}\mathsf{MSL} \simeq \mathsf{MSO}, \qquad r_\mathbb{R}\mathsf{MSp} \simeq \mathsf{MU}.$$

Proposition B (Propositions 3.12 and 3.13)**.** *Let $\mathsf{KW} := \mathsf{KO}[\eta^{-1}]$ be the motivic spectrum representing Balmer-Witt K-theory (where η is the motivic Hopf map). There is an $\mathcal{E}_\infty$-map of spectra*

$$\gamma : \mathsf{L}(\mathbb{R}) \longrightarrow r_\mathbb{R}\mathsf{KW}$$

inducing an equivalence $\mathsf{L}(\mathbb{R})[1/2] \simeq r_\mathbb{R}\mathsf{KW}$. In particular, there are equivalences of $\mathcal{E}_\infty$-rings

$$\mathsf{KO}^{\mathsf{top}}[1/2] \simeq r_\mathbb{R}\mathsf{KW} \simeq r_\mathbb{R}\mathsf{KO}.$$

The proof of Proposition B relies crucially on the description of the real realization functor as a localization away from $\rho : \mathcal{S}^0 \to \mathbb{G}_m$ ([Bac18, Thm. 35], see Theorem 1.13), and the relation $\rho\eta \simeq -2$ in $\mathsf{SH}(\mathbb{R})[\rho^{-1}]$.

Theorem A is obtained as a corollary of a more general result about Thom spectrum functors. Indeed, the motivic spectra MGL, MSL, and MSp can be described as elements in the image of the motivic Thom spectrum functor, whose symmetric monoidal structure is used to endow them with an $\mathcal{E}_\infty$-structure (see [BH21, §16.2 and Ex. 16.22], [BH20, below Lemma 4.6]). Similarly, MO, MSO, and MU are endowed with the structure of $\mathcal{E}_\infty$-rings by expressing them as (topological) Thom spectra [Bea23]. We will show that the motivic and topological Thom spectrum functors correspond to each other under real realization, as symmetric monoidal functors. To give a more precise description, we first consider the assignment $X \in \mathsf{Sm}_{\mathbb{R}} \mapsto \mathsf{Pic}(\mathsf{SH}(X))$ (where $\mathsf{Pic}(\mathsf{SH}(X)) \subseteq \mathsf{SH}(X)$ is the groupoid of invertible objects with respect to the smash product) as a presheaf of categories on smooth schemes of finite type over $\mathbb{R}$, with functoriality given by pullback. The motivic Thom spectrum functor takes as input a presheaf $\mathcal{F}$ of spaces over $\mathsf{Sm}_{\mathbb{R}}$, together with a morphism of presheaves $\mathcal{F} \to \mathsf{Pic}(\mathsf{SH}(-))$, and produces a motivic spectrum over $\mathbb{R}$ out of it. We can then take the real realization of such a motivic spectra. But one could also first apply a functor induced by real realization to the morphism $\mathcal{F} \to \mathsf{Pic}(\mathsf{SH}(-))$, to obtain a map of spaces $r_{\mathbb{R}}(\mathcal{F}) \to \mathsf{Pic}(\mathsf{Sp})$, where $\mathsf{Pic}(\mathsf{Sp})$ is the groupoid of invertible objects in spectra, and then apply the topological Thom spectrum functor to this map. Both constructions yield a spectrum, and even an $\mathcal{E}_\infty$-ring if the arrow $\mathcal{F} \to \mathsf{Pic}(\mathsf{SH}(-))$ was a morphism of commutative algebras to begin with. We show that these constructions agree. More generally, we will consider morphism of presheaves $\mathcal{F} \to \mathsf{SH}(-)^{\omega,\simeq}$ where $\mathsf{SH}(X)^{\omega,\simeq}$ is the maximal subgroupoid in the full subcategory of compact objects $\mathsf{SH}(X)^\omega \subseteq \mathsf{SH}(X)$. The precise statement reads as follows:

Theorem C (Theorem 2.44)**.** *The real realization of the motivic multiplicative Thom spectrum functor $M : \mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\omega,\simeq}} \to \mathsf{SH}$ (constructed in [BH21, §16.3]) is equivalent as a symmetric monoidal functor to the topological multiplicative Thom spectrum functor $M_{\mathsf{top}} : \mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}} \to \mathsf{Sp}$ (constructed in [ABG18, Thm. 1.6]). More precisely, there is a commutative diagram of symmetric monoidal functors (all notations will be defined in Section 2)*

$$\begin{array}{ccc} \mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\omega,\simeq}} & \xrightarrow{r_{\mathbb{R}} \circ \alpha_\sharp} & \mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}} \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M_{\mathsf{top}}} \\ \mathsf{SH}(\mathbb{R}) & \xrightarrow[r_{\mathbb{R}}]{} & \mathsf{Sp}. \end{array}$$

The motivic Thom spectrum functor is an instance of the more general motivic colimit functors [BEH22], defined for SH replaced by another presheaf $\mathcal{F}$. To prove Theorem C, we establish a naturality statement for this construction in the presheaf $\mathcal{F}$. Using this result, we will compare the motivic Thom spectrum functor to another motivic colimit functor $M_{\mathcal{R}}$ related to real realization, which takes its values in Sp. Finally, the description of the unstable real realization functor as localization [ABEH25] will imply that $M_{\mathcal{R}}$ factors through $\mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}}$ (which is the source category of the topological Thom spectrum functor M_{top}) via real realization, and that the functor obtained in this way is indeed equivalent to M_{top}. This gives us the commutative square in Theorem C.

Theorems A and B can in turn be used to compute the real realization of a refinement of KO, namely its very effective cover ko. Very effectiveness is a certain notion of "connectivity" for motivic spectra. The very effective cover functor being lax symmetric monoidal, ko inherits an $\mathcal{E}_\infty$-structure from that of KO, and so does its real realization. We first compute the homotopy ring of $r_{\mathbb{R}}\mathsf{ko}$, by relating ko to other motivic spectra whose real realizations can be computed more easily, see below. This homotopy ring is a polynomial ring over $\mathbb{Z}$ with a single generator x in degree 4, $\mathbb{Z}[x]$. We will then identify the underlying $\mathcal{E}_1$-ring, by describing it as an explicit pullback square. Indeed, in general, if p is a prime, any spectrum or $\mathcal{E}_n$-ring can be expressed as a pullback of its localizations away from p and at (p), respectively, over its rationalization. Such a square may be called a p-local fracture square. Theorem B and its proof advocate in the favor of inverting 2. Our strategy will then be to identify explicitly the $\mathcal{E}_1$-rings and maps appearing in the 2-local fracture square for $r_{\mathbb{R}}\mathsf{ko}$. The localization away from 2 will be computed using Theorem B, and the localization at (2) will be identified with a free $\mathcal{E}_1$-algebra in the category of $\mathsf{H}\mathbb{Z}_{(2)}$-modules. In particular, this will require us to produce an $\mathcal{E}_1$-map of $\mathsf{H}\mathbb{Z}_{(2)}$-modules $\mathsf{H}\mathbb{Z}_{(2)} \to (r_{\mathbb{R}}\mathsf{ko})_{(2)}$, and we will also use our computation of the homotopy ring of $r_{\mathbb{R}}\mathsf{ko}$. The existence of $\mathcal{E}_2$-maps $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{MSO}_{(2)}$ in Sp and $\mathsf{MSL} \to \mathsf{ko}$ in $\mathsf{SH}(\mathbb{R})$, respectively, has been shown in the past ([HLN21, Cor. 3.7] and [HJNY22, Thm. 10.1] respectively). Using Theorem A, which implies in particular that $r_{\mathbb{R}}(\mathsf{MSL}) \simeq \mathsf{MSO}$ as $\mathcal{E}_2$-rings, we will finally obtain an $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}_{(2)} \to (r_{\mathbb{R}}\mathsf{ko})_{(2)}$,

sufficient for our purposes. Combining these results with a careful analysis of the equivalences we have chosen yields the following 2-local fracture square for $r_{\mathbb{R}}\mathsf{ko}$ as an $\mathcal{E}_1$-ring.

Theorem D (Theorem 4.8)**.** *There is a Cartesian square of $\mathcal{E}_1$-rings (and thus also of spectra)*

$$\begin{array}{ccc} r_{\mathbb{R}}\mathsf{ko} & \xrightarrow{x\mapsto\beta_4/2} & \mathsf{KO}^{\mathsf{top}}_{\geq 0}[1/2] \\ {\scriptstyle x\mapsto t^4}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{ch}\ \ \beta_4/2\mapsto t^4} \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] & \xrightarrow[t^4\mapsto t^4]{} & \mathsf{H}\mathbb{Q}[t^4]. \end{array}$$

In particular, there is an isomorphism of graded rings $\pi_*(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}[x]$ where x has degree 4.

Here, the assignments of elements labeling the arrows describe the action of the maps on the fourth homotopy groups; $\mathsf{KO}^{\mathsf{top}}$ is the topological spectrum representing real topological K-theory; and $\mathsf{H}\mathbb{Q}[t^4]$ is the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebra on one generator t^4 in degree 4. It is an $\mathsf{H}\mathbb{Q}$-module, with homotopy ring a polynomial ring over $\mathbb{Q}$ with a single generator in degree 4. The definition of $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$ is similar.

Hebestreit, Land, and Nikolaus exhibited an identical fracture square for the connective L-theory of the real numbers [HLN21, p3]. Comparing the two results proves:

Theorem E (Theorem 4.17)**.** *There is an equivalence of $\mathcal{E}_1$-rings*

$$r_{\mathbb{R}}\mathsf{ko} \simeq \mathsf{L}(\mathbb{R})_{\geq 0}$$

between the real realization of the very effective Hermitian K-theory spectrum and the connective cover of the L-theory spectrum of $\mathbb{R}$.

As mentioned above, the proof of Theorem D uses the relation of ko to other motivic spectra whose real realization is easier to compute (in particular to determine the homotopy ring of $r_{\mathbb{R}}\mathsf{ko}$, and to analyze the maps appearing in the fracture square). Namely, ko is related to kgl, the very effective cover of algebraic K-theory KGL, via a cofiber sequence [ARØ20, Prop. 2.11]

$$\Sigma^{1,1}\mathsf{ko} \xrightarrow{\eta} \mathsf{ko} \longrightarrow \mathsf{kgl}$$

where η is the motivic Hopf map. We will also use the decomposition of ko into easier pieces, that is, its very effective slices, which have been computed in [Bac17], and involve in particular the motivic cohomology spectra $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$, and their variant $\widetilde{\mathsf{H}\mathbb{Z}}$. We state the relevant results below. Although they are not new, they may not appear in this form or with these proofs in the literature.

Proposition F (Proposition 3.6)**.** *Let $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$ be the motivic spectra representing motivic cohomology with $\mathbb{Z}$-coefficients and $\mathbb{Z}/2$-coefficients, respectively. Then, there is an equivalence of $\mathcal{E}_1$-maps*

$$r_{\mathbb{R}}(\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2) \simeq (\mathsf{H}\mathbb{Z}/2[t^2] \to \mathsf{H}\mathbb{Z}/2[t])$$

where $\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$ is the projection, and $\mathsf{H}\mathbb{Z}/2[t^2] \to \mathsf{H}\mathbb{Z}/2[t]$ is the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras (see Definition 1.20) inducing the inclusion $\mathbb{Z}/2[t^2] \subseteq \mathbb{Z}/2[t]$ in homotopy.

Proposition G (Proposition 3.8)**.** *Let KGL be the motivic spectrum representing algebraic K-theory, and kgl be its very effective cover. Then, there is an equivalence of $\mathcal{E}_1$-maps*

$$r_{\mathbb{R}}(\mathsf{kgl} \to \mathsf{H}\mathbb{Z}) \simeq (\mathsf{H}\mathbb{Z}/2[t^4] \to \mathsf{H}\mathbb{Z}/2[t^2])$$

where $\mathsf{kgl} \to \mathsf{kgl}/\beta_{\mathsf{KGL}} \simeq \mathsf{H}\mathbb{Z}$ is the projection (with β_{KGL} the periodicity generator for KGL), and $\mathsf{H}\mathbb{Z}/2[t^4] \to \mathsf{H}\mathbb{Z}/2[t^2]$ is the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras (see Definition 1.20) inducing the inclusion $\mathbb{Z}/2[t^4] \subseteq \mathbb{Z}/2[t^2]$ in homotopy.

Proposition H (Proposition 3.11)**.** *On the homotopy rings, the real realization of the quotient map*

$$\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}^{MW}_* \longrightarrow f_0(\underline{K}^{MW}_*/\eta) = f_0\underline{K}^{M}_* = \mathsf{H}\mathbb{Z}$$

identifies with the quotient map $\mathbb{Z}[t^2]/(2t^2) \to \mathbb{Z}[t^2]/2$.

Organization of the article. In Section 1, we make some recollections about the notions of (very) effectiveness for motivic spectra, the real realization functor, and free $\mathcal{E}_1$-HA-algebras in topological spectra. It plays the role of a toolbox whose results will be cited in the remainder of the paper. Section 2 is dedicated to the comparison of the motivic and topological Thom spectrum functors, i.e., to the proofs of Theorems A and C. In Section 3, we compute the real realizations of several classical motivic spectra, in particular we prove Propositions F, G, and H, together with Theorem B. These computations will come in handy in Section 4, where we compute $r_{\mathbb{R}}\mathsf{ko}$ and prove Theorems D and E. We finish off with two appendices A and B about the Day convolution symmetric monoidal structure on categories of presheaves; and symmetric monoidal structures on slice categories respectively. At the very end is an Index of notation, to which the reader may refer for any unexplained notation.

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Conventions and prerequisites

- For introductory material regarding the topics discussed in this article, the reader may refer to [Ban25].
- We work in the ∞-categorical setting, following [Lur09] and [Lur17]. We simply say "category" for "∞-category", and will specify "1-category" when relevant. Let Cat_∞ be the ∞-category of small ∞-categories.
- We denote by Spc the category of spaces and by Sp the category of spectra.
- Unless mentioned otherwise, k denotes a fixed base field, and S a fixed qcqs base scheme. When working with Hermitian K-theory, we assume $\mathsf{char}(k) \neq 2$. Let Sm_S be the category of smooth schemes of finite type over S. We denote by $\mathsf{Spc}(S)$ and $\mathsf{SH}(S)$ the categories of motivic spaces and motivic spectra, respectively, over S (constructed in the setting of model categories in [MV99], and with ∞-categories in [Rob13] for example). For a smooth scheme X over S or a space Y, we will often also denote by X and Y the corresponding motivic spaces or motivic spectra. Let T be the infinite suspension spectrum of the pointed motivic space $(\mathbb{P}^1, \infty)$.
- We denote by $r_{\mathbb{R}} : \mathsf{SH}(\mathbb{R}) \to \mathsf{Sp}$ the real realization functor (see [Bac18, §10] for example). We also denote the corresponding functors $\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) \to \mathsf{Spc}$ and $\mathsf{Spc}(\mathbb{R}) \to \mathsf{Spc}$, or their pointed analogs, by $r_{\mathbb{R}}$.
- For p a prime, *localization at* (p) means inverting every prime different from p, whereas *localization away from* p means inverting p.

1. A few preliminaries

This introductory section contains some preliminaries about the notions of effectiveness and very effectiveness for motivic spectra (Subsection 1.1), the real realization functor (Subsection 1.2), and free $\mathcal{E}_1$-HA-algebras (in topological spectra), for A a discrete commutative ring and HA its Eilenberg-Mac Lane spectrum (Subsection 1.3). The latter will appear in our computations in Sections 3 and 4.

1.1. Recollections about (very) effectiveness of motivic spectra.

In this subsection, we recall the definitions of the notions of effectiveness [Voe02, §2] and very effectiveness [SØ12, Def. 5.5], essentially to fix notation. The upshot is that these are two different notions of connectivity for motivic spectra. For some justification of these concepts, we refer to the introduction of [Bac17].

Definition 1.1 ([BH21, §13])**.** The *category of effective motivic spectra* $\mathsf{SH}(k)^{\mathsf{eff}}$ is the subcategory of $\mathsf{SH}(k)$ generated under colimits by all $\mathcal{S}^n \wedge \Sigma^\infty_+ X$ for $X \in \mathsf{Sm}_k$ and $n \in \mathbb{Z}$. The *category of very effective motivic spectra* $\mathsf{SH}(k)^{\mathsf{veff}}$ is the subcategory of $\mathsf{SH}(k)$ generated under colimits by all $\mathcal{S}^n \wedge \Sigma^\infty_+ X$ for $X \in \mathsf{Sm}_k$ and $n \geq 0$. For $n \geq 0$, let $\mathsf{SH}(k)^{\mathsf{eff}}(n) := T^{\wedge n} \wedge \mathsf{SH}(k)^{\mathsf{eff}}$ be the category of *n-effective spectra* and $\mathsf{SH}(k)^{\mathsf{veff}}(n) := T^{\wedge n} \wedge \mathsf{SH}(k)^{\mathsf{veff}}$ be the category of *very n-effective spectra*.

Definition 1.2 ([Mor99, Thm. 5.2.3], [BH21, App. B])**.** The *homotopy t-structure* on $\mathsf{SH}(k)$ is defined by

$$\mathsf{SH}(k)_{\geq 0} = \{E \in \mathsf{SH}(k) \mid \underline{\pi}_i(E)_* = 0 \ \forall i < 0\},$$
$$\mathsf{SH}(k)_{\leq 0} = \{E \in \mathsf{SH}(k) \mid \underline{\pi}_i(E)_* = 0 \ \forall i > 0\}.$$

Equivalently, $\mathsf{SH}(k)_{\geq 0}$ is generated under colimits and extensions by $\{\Sigma^{p,q}\Sigma^\infty_+ X \mid X \in \mathsf{Sm}_k, p-q \geq 0\}$.

Remark 1.3. The smash product symmetric monoidal structure on $\mathsf{SH}(k)$ is compatible with the homotopy t-structure in the sense of [AN21, Def. A.10]. To see this, we use the equivalent characterization of the connective part in Definition 1.2. Then the unit is one of the generators of the connective part. Moreover, the tensor product functor preserves colimits in each variable; since $\mathsf{SH}(k)$ is stable, this functor is in particular exact in both variables. Therefore, to check that $\mathsf{SH}(k)_{\geq 0}$ is stable under tensor products we only have to show that the aforementioned collection of generators also is. For any $X, Y \in \mathsf{Sm}_k$ and $p, q, p', q' \in \mathbb{Z}$ with $p - q \geq 0$ and $p' - q' \geq 0$, we have $\Sigma^{p,q}\Sigma^\infty_+ X \wedge \Sigma^{p',q'}\Sigma^\infty_+ Y \simeq \Sigma^{p+p',q+q'}\Sigma^\infty_+(X \times Y)$ with $p + p' - (q + q') \geq 0$, which is again in the connective part. In particular, $\mathsf{SH}(k)_{\geq 0}$ is a symmetric monoidal subcategory of $\mathsf{SH}(k)$, and there is a unique symmetric monoidal structure on the heart $\mathsf{SH}(k)^\heartsuit$ such that the truncation $\mathsf{SH}(k)_{\geq 0} \to \mathsf{SH}(k)^\heartsuit$ is symmetric monoidal [AN21, Lem. A.12].

Definition 1.4 (e.g. [Bac17, Prop. 4])**.** The *effective homotopy t-structure* on $\mathsf{SH}(k)^{\mathsf{eff}}$ is defined by

$$\mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0} = \{E \in \mathsf{SH}(k)^{\mathsf{eff}} \mid \underline{\pi}_i(E)_0 = 0 \ \forall i < 0\},$$
$$\mathsf{SH}(k)^{\mathsf{eff}}_{\leq 0} = \{E \in \mathsf{SH}(k)^{\mathsf{eff}} \mid \underline{\pi}_i(E)_0 = 0 \ \forall i > 0\}.$$

For $n \in \mathbb{Z}$, let $\tau_{\geq_e n}$ and $\tau_{\leq_e n}$ be the truncation functors with respect to this t-structure.

Proposition 1.5.

(i) We have $\mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0} = \mathsf{SH}(k)^{\mathsf{veff}}$.

(ii) The smash product symmetric monoidal structure on $\mathsf{SH}(k)$ *restricts to symmetric monoidal structures on both* $\mathsf{SH}(k)^{\mathsf{eff}}$ *and* $\mathsf{SH}(k)^{\mathsf{veff}}$*, such that the inclusion functors are symmetric monoidal.*

Proof. Part (i) appears in [Bac17, Prop. 4]. For part (ii), note that since these two subcategories are full and contain the unit, by [Lur17, Prop. 2.2.1.1 and Rmk. 2.2.1.2] we only have to show that they are stable under the smash product. The latter preserves colimits in both variables, so we only have to check the statement for generators. For all $m, n \in \mathbb{Z}$ and $X, Y \in \mathsf{Sm}_k$, we have

$$(\mathcal{S}^n \wedge \Sigma^\infty_+ X) \wedge (\mathcal{S}^m \wedge \Sigma^\infty_+ Y) \simeq \mathcal{S}^{m+n} \wedge \Sigma^\infty(X_+ \wedge Y_+) \simeq \mathcal{S}^{m+n} \wedge \Sigma^\infty_+(X \times Y),$$

which is by definition effective, and even very effective when $n, m \in \mathbb{N}$. □

Lemma 1.6. *The inclusion functors* $\iota_n : \mathsf{SH}(k)^{\mathsf{eff}}(n) \hookrightarrow \mathsf{SH}(k)$ *and* $\tilde{\iota}_n : \mathsf{SH}(k)^{\mathsf{veff}}(n) \hookrightarrow \mathsf{SH}(k)$ *admit right adjoints* r_n *and* $\tilde{r}_n$ *respectively, for all* $n \geq 0$.

Proof. This follows from the adjoint functor theorem [Lur09, Cor. 5.5.2.9] and presentability of the categories involved [Hoy14, Prop. C.12.(2)]. □

Definition 1.7. We call the compositions $\iota_n \circ r_n =: f_n$ and $\tilde{\iota}_n \circ \tilde{r}_n =: \tilde{f}_n$ the *n-effective cover* and *very n-effective cover* functors respectively. For all $E \in \mathsf{SH}(k)$, we obtain by adjunction natural maps $f_{n+1}E \to f_n E$ and $\tilde{f}_{n+1}E \to \tilde{f}_n E$ for all $n \geq 0$. Their cofibers $s_n E := \mathsf{cof}(f_{n+1}E \to f_n E)$ and $\tilde{s}_n E := \tilde{f}_{n+1} \to \tilde{f}_n E$ are called the *n-effective slice* and *very n-effective slice* of E, respectively. By *(very) effective cover*, we mean the (very) 0-effective cover.

Proposition 1.8. *The functor* $\iota_0 : \mathsf{SH}(k)^{\mathsf{eff}} \to \mathsf{SH}(k)$ *is right t-exact, and* $r_0 : \mathsf{SH}(k) \to \mathsf{SH}(k)^{\mathsf{eff}}$ *is t-exact and lax symmetric monoidal. It restricts to a lax symmetric monoidal functor* $r_0^\heartsuit : \mathsf{SH}(k)^\heartsuit \to \mathsf{SH}(k)^{\mathsf{eff},\heartsuit}$.

Proof. The claims about t-exactness are proven in [Bac17, Prop. 4.(3)]. As a right adjoint to the inclusion, which is symmetric monoidal (Proposition 1.5(ii)), r_0 is also lax symmetric monoidal by the doctrinal adjunction principle (follows from of [Lur17, Cor. 7.3.2.7]). As a t-exact functor, r_0 carries the non-negative part to the non-negative part of the t-structures, and the heart to the heart. Its (co)restriction to the symmetric monoidal subcategories $\mathsf{SH}(k)_{\geq 0}$ and $\mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0}$ is then also lax symmetric monoidal. Its (co)restriction to the hearts $r_0^\heartsuit$ can be written as the composition

$$\mathsf{SH}(k)^\heartsuit \hookrightarrow \mathsf{SH}(k)_{\geq 0} \xrightarrow{r_0} \mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0} \xrightarrow{\pi_0} \mathsf{SH}(k)^{\mathsf{eff},\heartsuit}.$$

Here, π_0 is symmetric monoidal by definition of the tensor product on the heart [AN21, Lem. A.12]. The first inclusion is lax symmetric monoidal as a right adjoint to the truncation, which is symmetric monoidal by Remark 1.3. Thus, $r_0^\heartsuit$ is lax symmetric monoidal. □

The (very) n-effective covers and slices behave well with respect to $\mathbb{P}^1$-suspension:

Proposition 1.9 ([Bac17, Lem. 8])**.** *For all $E \in \mathsf{SH}(k)$ and $n \geq 0$, we have $f_n E \wedge T \simeq f_{n+1}(E \wedge T)$. Similar formulas hold for $\tilde{f}_n$, s_n and $\tilde{s}_n$.*

1.2. **Recollections about the real realization functor.**

We now state several results about the real realization functor, recalling its construction in the process. In this subsection, we assume that the field k comes with a fixed embedding $k \hookrightarrow \mathbb{R}$. Considering the case $k = \mathbb{R}$ is not really a loss of generality; the real realization functor over the base k is equivalent to the precomposition of the real realization functor over the base $\mathbb{R}$ by the pullback along $\mathsf{Spec}(\mathbb{R}) \to \mathsf{Spec}(k)$.

Proposition 1.10. *The real realization functors $r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Spc}_*$ and $r_{\mathbb{R}} : \mathsf{SH}(k) \to \mathsf{Sp}$ are colimit-preserving, symmetric monoidal functors with respect to the smash products.*

Proof. This follows from the construction of $r_{\mathbb{R}}$, as we now recall (see [Bac18, §10] for example). The functor of real points $\mathsf{Sm}_k \to \mathsf{Spc}$ preserves finite products and thus left Kan extends to colimit-preserving, symmetric monoidal functors $\mathcal{P}(\mathsf{Sm}_k) \to \mathsf{Spc}$ for the Cartesian structures, and $\mathcal{P}(\mathsf{Sm}_k)_* \to \mathsf{Spc}_*$ for the smash products. Passing to the localization, we obtain a colimit preserving, symmetric monoidal functor $r_{\mathbb{R}} : \mathsf{Spc}(k)_* \to \mathsf{Spc}_*$. The postcomposition $\mathsf{Spc}(k)_* \to \mathsf{Spc}_* \to \mathsf{Sp}$ by the infinite suspension functor inverts $\mathbb{P}^1$, and thus lifts to a colimit preserving, symmetric monoidal functor $r_{\mathbb{R}} : \mathsf{SH}(k) \to \mathsf{Sp}$. □

Here is one example of why very effectiveness is a reasonable notion of connectivity in $\mathsf{SH}(k)$.

Lemma 1.11. *For all $m \geq 0$, the restriction of $r_{\mathbb{R}}$ to $\mathsf{SH}(k)^{\mathsf{veff}}(m)$ takes values in m-connective spectra.*

Proof. Since $\mathsf{SH}(k)^{\mathsf{veff}}(m) = T^{\wedge m} \wedge \mathsf{SH}(k)^{\mathsf{veff}}$ by definition and $r_{\mathbb{R}}(T^{\wedge m}) = \mathbb{S}^m$, it suffices to prove the case $m = 0$. By definition, $\mathsf{SH}(k)^{\mathsf{veff}}$ is generated under colimits by all objects of the form $\mathcal{S}^n \wedge \Sigma^\infty_+ X$ for $n \geq 0$ and $X \in \mathsf{Sm}_k$. These all realize to connective spectra because $r_{\mathbb{R}}(\mathcal{S}^n \wedge \Sigma^\infty_+ X) = \mathbb{S}^n \wedge \Sigma^\infty_+ r_{\mathbb{R}}(X)$ is connective for all $n \geq 0$ and $X \in \mathsf{Sm}_k$. This concludes because $r_{\mathbb{R}}$ commutes with colimits, and a colimit of connective spectra is connective (this holds for the non-negative part of any t-structure). □

The category of spectra Sp is obtained as a localization of a category of *prespectra*: if one views a spectrum as a sequence of pointed spaces $(c_n)_{n\in\mathbb{N}}$ with equivalences $c_n \to \Omega c_{n+1}$, then prespectra are defined as similar sequences for which the structure maps are not required to be equivalences. More precisely, the category of prespectra is that of reduced functors $\mathsf{Spc}_*^{\mathsf{fin}} \to \mathsf{Spc}$, and Sp is the full subcategory of reduced excisive functors [Lur17, Prop. 1.4.2.13.]. The latter is in particular a localization of the category of prespectra [Lur17, Rmk. 1.4.2.4.], and the localization functor is called *spectrification*.

Lemma 1.12. *Let $E = (E_0, E_1, \dots) \in \mathsf{SH}(k)$. Then, the real realization of E is computed as the spectrification of the prespectrum $(r_{\mathbb{R}}(E_n))_{n\in\mathbb{N}}$. More explicitly, for all $n \in \mathbb{Z}$,*

$$r_{\mathbb{R}}(E)_n \simeq \mathsf{colim}_m \Omega^m r_{\mathbb{R}}(E_{n+m}).$$

In particular, $\pi_n(r_{\mathbb{R}}(E)) \cong \mathsf{colim}_m \pi_{n+m}(r_{\mathbb{R}}(E_m))$.

Proof. First note that $(r_{\mathbb{R}}(E_n))_{n\in\mathbb{N}}$ is a prespectrum via the structure maps obtained by real realization of those of E. Indeed, a map $T \wedge E_n \to E_{n+1}$ realizes to a map $\mathbb{S}^1 \wedge r_{\mathbb{R}}(E_n) \to r_{\mathbb{R}}(E_{n+1})$. The spectrification functor for topological prespectra sends a sequence $(E_n)_{n\in\mathbb{N}}$ to $\mathsf{colim}_n \Sigma^{-n}\Sigma^\infty E_n$ where the maps in the colimit are induced by the structure maps $\Sigma E_n \to E_{n+1}$ [Sch97, Lem. 2.1.3]. Similarly, for motivic prespectra, spectrification sends a sequence $(E_n)_{n\in\mathbb{N}}$ to $\mathsf{colim}_n \Sigma_T^{-n}\Sigma^\infty_T E_n$ with maps induced by $\Sigma_T E_n \to E_{n+1}$. If E as in the statement is already a spectrum, it is its own spectrification, and therefore we have

$$r_{\mathbb{R}}(E) \simeq r_{\mathbb{R}}(\mathsf{colim}_n(T^{\wedge(-n)} \wedge \Sigma^\infty_T E_n)) \simeq \mathsf{colim}_n(\mathbb{S}^{\wedge(-n)} \wedge \Sigma^\infty_{\mathbb{S}^1} r_{\mathbb{R}}(E_n)).$$

The right-hand side is exactly the spectrification of the sequence $(r_{\mathbb{R}}(E_n))_{n\in\mathbb{N}}$ with structure maps given by the real realizations of those of E. For the last claim, we have

$$\begin{aligned}\pi_n(r_{\mathbb{R}}(E)) &\cong \mathsf{colim}_m \pi_{n+m}(r_{\mathbb{R}}(E)_m) \cong \mathsf{colim}_m \pi_{n+m}(\mathsf{colim}_k \Omega^k r_{\mathbb{R}}(E_{m+k})) \\ &\cong \mathsf{colim}_m \mathsf{colim}_k \pi_{n+m+k}(r_{\mathbb{R}}(E_{m+k})) \cong \mathsf{colim}_\ell \pi_{\ell+n}(r_{\mathbb{R}}(E_\ell)).\end{aligned}$$

□

Theorem 1.13 ([Bac18, Thm. 35])**.** *Let S be a qcqs scheme, and $\rho : \mathcal{S}^0 \to \mathbb{G}_m$ in $\mathsf{SH}(S)$ be the inclusion on the points 1 and -1. There is an equivalence of categories*

$$\mathsf{SH}(S)[\rho^{-1}] \xrightarrow{\simeq} \mathsf{SH}(\mathsf{Shv}(RS))$$

between the category of ρ-local motivic spectra over S, and the category of spectra associated with the topos of sheaves of spaces on RS, the real spectrum of S [Sch94, 0.4.2]. In particular, there is an equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathsf{Sp}$ such that the composite

$$\mathsf{SH}(\mathbb{R}) \longrightarrow \mathsf{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathsf{Sp}$$

is naturally equivalent to the real realization functor $r_{\mathbb{R}}$.

Lemma 1.14. *For any $E \in \mathsf{SH}(\mathbb{R})$, the homotopy groups of its real realization can be computed as*

$$\forall m \in \mathbb{Z}, \quad \pi_m(r_{\mathbb{R}}(E)) = \mathsf{colim}_n(\cdots \longrightarrow \underline{\pi}_m(E)_n(\mathsf{Spec}(\mathbb{R})) \xrightarrow{\rho} \underline{\pi}_m(E)_{n+1}(\mathsf{Spec}(\mathbb{R})) \longrightarrow \cdots).$$

Proof. Call F the equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \to \mathsf{Sp}$ above, and $L_\rho : \mathsf{SH}(\mathbb{R}) \to \mathsf{SH}(\mathbb{R})[\rho^{-1}]$ the canonical localization functor. Then, for all $m \in \mathbb{Z}$ and $E \in \mathsf{SH}(\mathbb{R})$, we have

$$\begin{aligned}
\pi_m(r_{\mathbb{R}}E) &\cong [\mathbb{S}^m, r_{\mathbb{R}}E] \cong [r_{\mathbb{R}}(\mathcal{S}^m), r_{\mathbb{R}}E] \cong [F(L_\rho(\mathcal{S}^m)), F(L_\rho(E))] \cong [L_\rho(\mathcal{S}^m), L_\rho(E)] \\
&\cong [\mathcal{S}^m, \mathsf{colim}_n E \wedge \mathbb{G}_m^{\wedge n}] && \text{(since } L_\rho \text{ is a left-adjoint)} \\
&\cong \mathsf{colim}_n [\mathcal{S}^m, E \wedge \mathbb{G}_m^{\wedge n}] && \text{(by compactness of } \mathcal{S}^m) \\
&\cong \mathsf{colim}_n \underline{\pi}_m(E)_n(\mathsf{Spec}(\mathbb{R}))
\end{aligned}$$

where the last isomorphism follows from the fact that the sections of the homotopy sheaves and presheaves of E agree on $\mathsf{Spec}(\mathbb{R})$. This holds because sheafification preserves stalks, and the functor of sections over $\mathsf{Spec}(\mathbb{R})$ is a stalk functor of the Nisnevich site on $\mathsf{Sm}_{\mathbb{R}}$ (by the description of the points of the Nisnevich site in [MV99, p99]). □

Lemma 1.15. *Let p be a prime. The real realization functor commutes with localization away from p, localization at (p), and rationalization.*

Proof. These operations are described by colimits of maps given by multiplication by an integer; both colimits and such maps are preserved by $r_{\mathbb{R}}$. Indeed, multiplication by an integer on some object X of a stable category is defined using the diagonal and codiagonal maps $X \to X^{\oplus p} \to X$. Since real realization is an exact functor, it preserves the latter. □

Bachmann's result about real realization of motivic spectra being a localization (Theorem 1.13) has an unstable counterpart. To describe it, we first need a definition.

Definition 1.16 ([Sch94, §1.2])**.** The real-étale topology on Sm_S is the topology generated by real-étale covers, that is, families $\{U_i \to X\}_{i \in I}$ of étale morphisms inducing a surjection $\coprod_{i \in I} R(U_i) \to R(X)$ (here $R(-)$ denotes the real spectrum functor appearing in Theorem 1.13).

Theorem 1.17 ([ABEH25, Thm. 1.1])**.** *The (unstable) real realization functor $r_{\mathbb{R}} : \mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) \to \mathsf{Spc}$ factors through the category $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))$ of $\mathbb{A}^1$-invariant real-étale sheaves, and the induced functor $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})) \to \mathsf{Spc}$, given by taking sections on $\mathsf{Spec}(\mathbb{R})$, is an equivalence*

$$\begin{array}{ccc}
\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) & \longrightarrow & \mathsf{Spc} \\
{\scriptstyle L_{\mathbb{A}^1,\mathsf{ret}}}\downarrow & \nearrow_{\simeq} & \\
L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})) & &
\end{array}$$

whose inverse is given by left Kan extension.

1.3. **Free $\mathcal{E}_1$-HA-algebras.**

In Sections 3 and 4, we will encounter $\mathcal{E}_\infty$-rings whose homotopy rings are polynomial in one variable. We will identify the underlying $\mathcal{E}_1$-structures as free $\mathcal{E}_1$-algebras in the category of HA-modules, where HA is the Eilenberg-Mac Lane spectrum associated with a discrete commutative ring A.

Definition 1.18. Let $R \in \mathsf{CAlg}(\mathsf{Sp})$. The *category of $\mathcal{E}_k$-R-algebras in* Sp is $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Mod}_R(\mathsf{Sp}))$, the category of $\mathcal{E}_k$-algebras in R-modules.

Proposition 1.19. *The forgetful functor* $U : \mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_R(\mathsf{Sp})) \to \mathsf{Sp}$ *admits a left adjoint* $F_{\mathcal{E}_1,R}$*, the* free $\mathcal{E}_1$-R-algebra functor*, given on the underlying spectra by* $E \mapsto \bigvee_{n\geq 0} E^{\wedge n} \wedge R$*.*

Proof. The functor U is the composition of two forgetful functors $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_R(\mathsf{Sp})) \to \mathsf{Mod}_R(\mathsf{Sp})$ and $\mathsf{Mod}_R(\mathsf{Sp}) \to \mathsf{Sp}$. Both admit left adjoints by [Lur17, Cor. 3.1.3.4] and [Lur17, Prop. 4.2.4.2], respectively; explicit expressions are given in [Lur17, Cor. 4.1.1.18] and [Lur17, Cor. 4.2.4.8],respectively. □

Definition 1.20. Let $n \geq 0$, and let A be a discrete commutative ring. Then $\mathsf{HA} \in \mathsf{CAlg}(\mathsf{Sp})$ and thus we may define $\mathsf{HA}[t^n] := F_{\mathcal{E}_1,\mathsf{HA}}(\Sigma^n\mathbb{S})$ the *free* $\mathcal{E}_1$-HA*-algebra over one generator in degree* n*.*

Proposition 1.21. *Let* $n \geq 0$*, and let* A *be a discrete commutative ring. The free* $\mathcal{E}_1$-HA*-algebra* $\mathsf{HA}[t^n]$ *has the following properties:*

(i) The underlying spectrum is given by $\bigvee_{j\geq 0} \Sigma^{nj}\mathsf{HA}$*.*

(ii) The homotopy rings is a polynomial ring on one generator in degree n*, i.e.,* $\pi_*(\mathsf{HA}[t^n]) \cong A[t^n]$*.*

(iii) Let E *be an* $\mathcal{E}_2$*-ring spectrum with an* $\mathcal{E}_2$*-map* $\mathsf{HA} \to E$*. Then* E *can be viewed as an* $\mathcal{E}_1$-HA*-algebra in a canonical way, and any element* $\alpha \in \pi_n(E)$ *determines a map* $\mathsf{HA}[t^n] \to E$ *in* $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_R(\mathsf{Sp}))$*, sending* t^n *to* α *in homotopy.*

Proof. Item (i) is given by Proposition 1.19. In particular, the computation of the homotopy groups follows. To finish the proof of item (ii), we must consider the ring structure. We have to show that multiplication by a generator of the n-th homotopy group induces isomorphisms in all degrees, i.e., that the composition

$$\mathbb{S}^n \wedge \mathsf{HA}[t^n] \xrightarrow{t^n\wedge \mathsf{id}} \mathsf{HA}[t^n] \wedge \mathsf{HA}[t^n] \longrightarrow \mathsf{HA}[t^n]$$

induces isomorphisms in homotopy in positive degrees. This map is given by the identification

$$\mathbb{S}^n \wedge \bigvee_{j\geq 0} \Sigma^{nj}\mathsf{HA} \simeq \bigvee_{j\geq 0} (\Sigma^n\mathbb{S} \wedge \Sigma^{nj}\mathsf{HA}) \longrightarrow \bigvee_{j\geq 0} \Sigma^{nj}\mathsf{HA}$$

mapping $\Sigma^n\mathbb{S} \wedge \Sigma^{nj}\mathsf{HA}$ to $\Sigma^{n(j+1)}\mathsf{HA}$ identically, so our claim follows. Item (iii), using that there is a forgetful functor $\mathsf{Alg}_{\mathcal{E}_2}(\mathsf{Sp})_{\mathsf{HA}/} \to \mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_{\mathsf{HA}}(\mathsf{Sp}))$ [ACB19, Rmk. 3.7], is simply rephrasing the fact that $F_{\mathcal{E}_1,\mathsf{HA}}$ is left adjoint to the forgetful functor. □

Remark 1.22. Proposition 1.21(iii) may seem a bit artificial. The point is that $\mathsf{HA}[t^n]$ is *not* a free object over $\mathbb{S}^n \in \mathsf{Sp}$ in the category $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Sp})_{\mathsf{HA}/}$; the latter category is a priori *not* equivalent to $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Mod}_{\mathsf{HA}}(\mathsf{Sp}))$ (see [Lur17, Warning 7.1.3.9] and [ACB19, Remark 3.7]). There is indeed a forgetful functor $\mathsf{Alg}_{\mathcal{E}_1}(\mathsf{Sp})_{\mathsf{HA}/} \to \mathsf{Mod}_{\mathsf{HA}}(\mathsf{Sp})$, but the $\mathcal{E}_1$-structure of an object on the left-hand side is not compatible with the HA-module structure obtained on the right-hand side. A higher commutativity (for instance E belonging to $\mathsf{Alg}_{\mathcal{E}_2}(\mathsf{Sp})_{\mathsf{HA}/}$ as in item (iii)) ensures this compatibility.

2. Motivic and topological multiplicative Thom spectra

In this section, our main goal is to show that the real realizations of the motivic $\mathcal{E}_\infty$-rings MGL, MSL, and MSp, representing, respectively, algebraic cobordism, special linear and symplectic algebraic cobordisms, are equivalent to the topological $\mathcal{E}_\infty$-rings MO, MSO, and MU, representing, respectively, cobordism, oriented and complex cobordisms. Again, this result was already proven at the level of spectra in [BH20, Cor. 4.7], but we want to take into account the $\mathcal{E}_\infty$-ring structures.

All six spectra are endowed with an $\mathcal{E}_\infty$-structure by expressing them as *Thom spectra*. Indeed, in both the motivic and topological settings, a symmetric monoidal Thom spectrum functor can be constructed (as we recall in Subsections 2.1 and 2.2, respectively) and these spectra are the images of some commutative algebra objects by this functor. Our result will thus be obtained as a corollary of the more general fact that the real realization of the motivic Thom spectrum functor agrees, as a symmetric monoidal functor, with the topological Thom spectrum functor. We will make this statement precise, and provide a proof, in Subsection 2.3.

2.1. The symmetric monoidal topological Thom spectrum functor.

Definition 2.1. Let Sp^ω be the full subcategory of Sp spanned by the compact objects, and let $\mathsf{Sp}^{\omega,\simeq}$ be the maximal (∞)-groupoid in the category Sp^ω. The (topological) *Thom spectrum functor*

$$M_{\mathsf{top}} : \mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}} \longrightarrow \mathsf{Sp}$$

sends an arrow $X \to \mathsf{Sp}^{\omega,\simeq}$ to the colimit of the diagram

$$X \longrightarrow \mathsf{Sp}^{\omega,\simeq} \hookrightarrow \mathsf{Sp}.$$

More precisely, under the equivalence $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}} \simeq \mathcal{P}(\mathsf{Sp}^{\simeq})$ from Lemma B.4 (at the level of categories), the functor M_{top} is the left Kan extension of the embedding $\mathsf{Sp}^{\omega,\simeq} \hookrightarrow \mathsf{Sp}$ (in particular, it preserves colimits).

Remark 2.2. In principle, one could consider for any arrow $X \to \mathsf{Sp}^{\simeq}$ the colimit of $X \to \mathsf{Sp}^{\simeq} \hookrightarrow \mathsf{Sp}$, and define a Thom spectrum functor $\mathsf{Spc}_{/\mathsf{Sp}^{\simeq}} \to \mathsf{Sp}$. This would however create some set-theoretic issues in our arguments, because $\mathsf{Sp}^{\simeq}$ is a large groupoid and thus cannot be viewed as an object in Spc. This is why we use instead the essentially small category Sp^{ω}. A more classical approach is to consider the even smaller subcategory $\mathsf{Pic}(\mathsf{Sp})$ of invertible objects in Sp (with respect to the tensor product). The examples of arrows $X \to \mathsf{Sp}^{\omega,\simeq}$ we will be interested in actually factor through $\mathsf{Pic}(\mathsf{Sp})$.

To upgrade M_{top} to a symmetric monoidal functor, we need in particular a symmetric monoidal structure on its source category. This is provided by Proposition B.3, if we choose a symmetric monoidal structure on Spc and an $\mathcal{E}_\infty$-algebra structure on $\mathsf{Sp}^{\omega,\simeq} \in \mathsf{Spc}$.

Lemma 2.3. *Consider the category of spaces with its Cartesian symmetric monoidal structure. Then* $\mathsf{Sp}^{\omega,\simeq} \in \mathsf{Spc}$ *inherits an* $\mathcal{E}_\infty$*-algebra structure from the symmetric monoidal structure of* Sp*.*

Proof. We will see in Proposition 2.33 that Sp^{ω} is an essentially small symmetric monoidal subcategory of Sp. Then, since the functor $(-)^{\simeq} : \mathsf{Cat}_\infty \to \mathsf{Gpd} \simeq \mathsf{Spc}$ is right adjoint to the forgetful functor, it preserves products, and thus is symmetric monoidal with respect to the Cartesian structure. Then, it carries symmetric monoidal category $\mathsf{Sp}^{\omega} \in \mathsf{CAlg}(\mathsf{Cat}_\infty^{\times})$ to an $\mathcal{E}_\infty$-space $\mathsf{Sp}^{\omega,\simeq} \in \mathsf{CAlg}(\mathsf{Spc}^{\times})$. □

Theorem 2.4 ([ABG18, Thm. 1.6])**.** *The topological Thom spectrum functor* M_{top} *admits the structure of a symmetric monoidal functor with respect to the symmetric monoidal structure on the slice category described in Proposition B.3 when* $\mathsf{Sp}^{\omega,\simeq}$ *has the* $\mathcal{E}_\infty$*-structure of Lemma 2.3 (equivalent by Lemma B.4 to the Day convolution structure) and the usual smash product of spectra. It is left Kan extended as a symmetric monoidal functor (in the sense of Definition A.4) from the inclusion* $\mathsf{Sp}^{\omega,\simeq} \hookrightarrow \mathsf{Sp}$*.*

The symmetric monoidal left Kan extension described in the theorem agrees with the construction in [ABG18, Thm. 1.6] because, by Proposition A.5, it is determined as a symmetric monoidal functor by its restriction to $*_{/\mathsf{Sp}^{\omega,\simeq}} \simeq \mathsf{Sp}^{\omega,\simeq}$.

Example 2.5 (see for instance [Bea23]). Let us see how one can define $\mathcal{E}_\infty$-structures on MO, MSO, and MU. The spectrum MO representing topological cobordism can be recovered as $M_{\mathsf{top}}(BO \to \mathsf{Sp}^{\omega,\simeq})$ where the map $BO \to \mathsf{Sp}^{\omega,\simeq}$ is the *j-homomorphism*. To define the latter, following [HY20, p. 2], one first considers the symmetric monoidal functor $\coprod_{n\in\mathbb{N}} BO_n \to \mathsf{Sp}$, where the left-hand side is viewed as the maximal ∞-groupoid in the 1-category of real vector spaces viewed as an ∞-category (or in other terms real vector bundles on the point), and a vector bundle is sent to the infinite suspension of its Thom space (in this case, the one-point compactification of our vector space). This map takes values in $\mathsf{Pic}(\mathsf{Sp})$, and thus factors through the group completion $\mathbb{Z}\times BO$ of $\coprod_{n\in\mathbb{N}} BO_n$. Restricting to $\{0\}\times BO$ (corresponding to rank 0 virtual vector bundles on the point), we obtain the j-homomorphism $BO \to \mathsf{Pic}(\mathsf{Sp}) \hookrightarrow \mathsf{Sp}^{\omega,\simeq}$. Similarly, we have $\mathsf{MSO} \simeq M_{\mathsf{top}}(BSO \to BO \to \mathsf{Sp}^{\omega,\simeq})$ and $\mathsf{MU} \simeq M_{\mathsf{top}}(BU \to BO \to \mathsf{Sp}^{\omega,\simeq})$. By Proposition B.5, any map of $\mathcal{E}_\infty$-spaces $X \to \mathsf{Sp}^{\omega,\simeq}$ defines a commutative algebra in $\mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}}$. We thus have $\mathcal{E}_\infty$-algebra structures on $BO \to \mathsf{Sp}^{\omega,\simeq}$, $BSO \to \mathsf{Sp}^{\omega,\simeq}$, and $BU \to \mathsf{Sp}^{\omega,\simeq}$ in the slice category (since the j-homomorphism is an $\mathcal{E}_\infty$-map by its construction as a symmetric monoidal functor). The Thom spectrum functor being symmetric monoidal, we obtain $\mathcal{E}_\infty$-structures on MO, MSO, and MU.

2.2. **The symmetric monoidal motivic Thom spectrum functor.**

In this subsection, we define the motivic analog to the topological Thom spectrum functor M_{top}. More precisely, this will be a symmetric monoidal functor $M_S : \mathcal{P}_\Sigma(\mathsf{Sm}_S)_{/\mathsf{SH}^{\omega,\simeq}} \to \mathsf{SH}(S)$, where $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$ is the category of spherical presheaves (Definition 2.12), and $\mathsf{SH}^{\omega,\simeq}$ is the presheaf sending $X \in \mathsf{Sm}_S$ to the maximal groupoid in the subcategory $\mathsf{SH}(X)^{\omega}$ of compact objects of $\mathsf{SH}(X)$. Again, the compactness requirement has only the purpose of avoiding set-theoretic issues.

2.2.1. *Functoriality of* $\mathsf{SH}(-)$ *and* $\mathsf{SH}(-)^\omega$.

The definition of the motivic Thom spectrum functor and its symmetric monoidal structure relies on the study of the functoriality in the base scheme S of the whole construction., in particular of the assignments $S \mapsto \mathsf{SH}(S)$ and $S \mapsto \mathsf{SH}(S)^\omega$. This is what this subsection is dedicated to. The functoriality properties of SH we will study fit into a complete six functors formalism, constructed by Ayoub [Ayo07]. We want to consider SH as a spherical presheaf (Definition 2.12) of (not necessarily small) symmetric monoidal categories on Sm_S.

Definition 2.6. Let fold be the collection of morphisms in Sm_S containing all finite coproducts of fold maps, i.e., maps of the form $Y := \coprod_{i\leq n} X_i^{\amalg m_i} \to \coprod_{i\leq n} X_i := X$ for $n \geq 0$ and $m_1, \dots, m_n \geq 0$.

Theorem 2.7 ([Ayo07, §1.4.1], [Hoy17])**.** *Let S be a qcqs scheme and consider a morphism $f : X \to Y$ in Sm_S.*

(i) For all $A, B \in \mathsf{Sm}_S$, we have $\mathsf{SH}(A \amalg B) \simeq \mathsf{SH}(A) \times \mathsf{SH}(B)$. If $\nabla \in \mathsf{fold}$ is of the form $V := \coprod_{i\leq n} U_i^{\amalg m_i} \to \coprod_{i\leq n} U_i := U$, let $\nabla_\otimes : \mathsf{SH}(V) \to \mathsf{SH}(U)$ be the map induced under these identifications by $\prod_{i\leq n} \mathsf{SH}(U_i)^{m_i} \to \prod_{i\leq n} \mathsf{SH}(U_i)$ given by the tensor products $\otimes : \mathsf{SH}(U_i)^{m_i} \to \mathsf{SH}(U_i)$ each $i \leq n$.

(ii) There are induced pullback-pushforward adjunctions

$$\begin{array}{ccccc} \mathcal{P}(\mathsf{Sm}_X) & \xrightarrow{\mathsf{L}_{\mathsf{mot}}} & \mathsf{Spc}(X) & \xrightarrow{\Sigma^\infty_+} & \mathsf{SH}(X) \\ {\scriptstyle f^*}\uparrow \dashv \downarrow{\scriptstyle f_*} & & {\scriptstyle f^*}\uparrow \dashv \downarrow{\scriptstyle f_*} & & {\scriptstyle f^*}\uparrow \dashv \downarrow{\scriptstyle f_*} \\ \mathcal{P}(\mathsf{Sm}_Y) & \xrightarrow{\mathsf{L}_{\mathsf{mot}}} & \mathsf{Spc}(Y) & \xrightarrow{\Sigma^\infty_+} & \mathsf{SH}(Y) \end{array}$$

where the squares involving f^* commute. The squares formed by f_* and the right adjoints $\iota \vdash \mathsf{L}_{\mathsf{mot}}$, $\Omega^\infty \vdash \Sigma^\infty_+$ also commute.

(iii) All three functors f^* in (ii) are symmetric monoidal, with respect to the Cartesian structure on the categories of presheaves and motivic spaces, and the smash product on the stable motivic categories.

(iv) If f is moreover smooth, there are induced adjunctions

$$\begin{array}{ccccc} \mathcal{P}(\mathsf{Sm}_X) & \xrightarrow{\mathsf{L}_{\mathsf{mot}}} & \mathsf{Spc}(X) & \xrightarrow{\Sigma^\infty_+} & \mathsf{SH}(X) \\ {\scriptstyle f_\sharp}\downarrow \dashv \uparrow{\scriptstyle f^*} & & {\scriptstyle f_\sharp}\downarrow \dashv \uparrow{\scriptstyle f^*} & & {\scriptstyle f_\sharp}\downarrow \dashv \uparrow{\scriptstyle f^*} \\ \mathcal{P}(\mathsf{Sm}_Y) & \xrightarrow{\mathsf{L}_{\mathsf{mot}}} & \mathsf{Spc}(Y) & \xrightarrow{\Sigma^\infty_+} & \mathsf{SH}(Y) \end{array}$$

where the squares involving either f^* or $f_\sharp$ commute.

(v) In the situation of (iv), the projection formula holds for all three functors $f_\sharp$, i.e., for every A in the source of $f_\sharp$ and B in its target, we have

$$f_\sharp(A \otimes f^*B) \simeq f_\sharp(A) \otimes B.$$

(vi) The constructions of (i) to (iv) are functorial in f.

(vii) For every commutative square in Sm_S

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f and f' smooth, there is an exchange transformation of functors $\mathsf{SH}(X) \to \mathsf{SH}(Y')$

$$\mathsf{Ex}^*_\sharp : f'_\sharp(g')^* \Longrightarrow g^* f_\sharp$$

(defined in the proof below). It is an equivalence if the square is Cartesian in Sm_S.

(viii) For every diagram in Sm_S of the form

$$\begin{array}{ccccc} W & \longleftarrow & Y' & \xrightarrow{\nabla'} & Z' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ X & \xleftarrow{f} & Y & \xrightarrow{\nabla} & Z \end{array}$$

with u and u' smooth morphisms and $\nabla, \nabla' \in \mathsf{fold}$, there is a distributivity transformation of functors $\mathsf{SH}(W) \to \mathsf{SH}(Z)$

$$\mathsf{Dis}_{\sharp\otimes} : u_\sharp \nabla'_\otimes (f')^* \Longrightarrow \nabla_\otimes (\pi_Y)_\sharp \pi_W^*$$

(defined in the proof below). It is an equivalence when the square on the right-hand side is a pullback and $Z' = R_{Y/Z}(W \times_X Y)$ is the Weil restriction (see [BH21, §2.3], and the proof below for an easier description in our case).

Proof. We give references for the proof; parts for which no reference is mentioned are proven in [Hoy17]. Part (i) is proven in [BH21, proof of Lem. 9.6]. In (vii), the exchange transformation is the composition

$$\mathsf{Ex}^*_\sharp : f'_\sharp(g')^* \xrightarrow{\eta_f} f'_\sharp(g')^* f^* f_\sharp \simeq f'_\sharp f'^* g^* f_\sharp \xrightarrow{\varepsilon_{f'}} g^* f_\sharp$$

where η_f and $\varepsilon_{f'}$ are the (co)units for the adjunctions $f_\sharp \dashv f^*$ and $f'_\sharp \dashv f'^*$, respectively, and the equivalence in the middle witnesses the commutativity of the square in (vii). It is an equivalence for pullback squares by [Hoy17, Cor. 6.12]. In $(viii)$, the distributivity transformation is the composition

$$\mathsf{Dis}_{\sharp\otimes} : u_\sharp \nabla'_\otimes(f')^* \xrightarrow{\mathsf{Ex}_{\sharp,\otimes}} \nabla_\otimes u'_\sharp (f')^* \simeq \nabla_\otimes (\pi_Y)_\sharp (f' \times u')_\sharp (f' \times u')^* \pi_W^* \xrightarrow{\varepsilon_{f' \times u'}} \nabla_\otimes (\pi_Y)_\sharp \pi_W^*$$

where the equivalence in the middle uses $u' \simeq \pi_Y \circ (f' \times u')$ and $f' \simeq \pi_W \circ (f' \times u')$, and $\mathsf{Ex}_{\sharp,\otimes}$ is the composition

$$\mathsf{Ex}_{\sharp,\otimes} : u_\sharp \nabla'_\otimes \xrightarrow{\eta_{u'}} u_\sharp \nabla'_\otimes (u')^* u'_\sharp \simeq u_\sharp u^* \nabla_\otimes u'_\sharp \xrightarrow{\varepsilon_u} \nabla_\otimes u'_\sharp$$

with the equivalence in the middle given by the symmetric monoidality of u^* (since the $\nabla_\otimes$ and $\nabla'_\otimes$ are given by the smash product). It is an equivalence when the conditions of statement $(viii)$ are satisfied by [BH21, Prop. 5.10]. The latter result assumes that h is quasi-projective, but we do not need it here: this assumption is only used to ensure the existence of the Weil restriction, however the latter always exists for maps in fold. Indeed, the Weil restriction $R_{Y/X}$ (when it exists) is defined by the property that $\mathsf{Sch}_X(U, R_{Y/X}V) \simeq \mathsf{Sch}_Y(U \times_X Y, V)$ for all X-scheme U and Y-scheme V. In the case of a map $(Y = \coprod_{i\leq n} X_i^{\amalg m_i} \to \coprod_{i\leq n} X_i = X) \in \mathsf{fold}$, it is easy to check that $R_{Y/X}V$ exists and is given by

$$R_{Y/X}V = \coprod_{i\leq n} \left(V_{X_{i,1}} \times_{X_i} \cdots \times_{X_i} V_{X_{i,m_i}}\right) \longrightarrow \coprod_{i\leq n} X_i,$$

where $X_{i,j}$ is the j-th component in $X_i^{\amalg m_i}$ and $V_{X_{i,j}}$ is the component of V over $X_{i,j}$, for all $i \leq n$ and $1 \leq j \leq m_i$. □

We now study the functoriality of the subpresheaf of categories of compact objects SH^ω, associating to each $X \in \mathsf{Sm}_{\mathbb{R}}$ the full subcategory of $\mathsf{SH}(X)$ spanned by its compact objects.

Theorem 2.8 ([Hoy14, Prop. C.12])**.** *Let S be a qcqs scheme. Then $\mathsf{SH}(S)$ is generated under colimits by the essentially small family of compact objects*

$$E_S = \{\Sigma^{-2n,-n}\Sigma^\infty_+ X \mid X \in \mathsf{Sm}_S, n \in \mathbb{N}\}.$$

Lemma 2.9. *Let $\mathcal{C}$ be a stable category generated under colimits by an essentially small family of compact objects $E \subseteq \mathcal{C}$. Then:*

(i) The full subcategory $\mathcal{C}^\omega$ of compact objects in $\mathcal{C}$ is the thick subcategory of $\mathcal{C}$ generated by E.

(ii) If $\mathcal{D}$ is a stable category with small colimits, and $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is an adjoint pair, then L preserves compact objects if and only if $L(E) \subseteq \mathcal{D}^\omega$.

Proof. Part (i) follows from [Nee92, Lem. 2.2]. In part (ii), the "only if" direction is tautological. For the converse, note that since L preserves colimits, the full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ of objects c' such that $L(c') \in \mathcal{D}^\omega$ is thick, and contains E by assumption. Using part (i), $\mathcal{C}'$ contains $\mathcal{C}^\omega$, as claimed. □

Lemma 2.10. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be finitely many categories. Then, the subcategory of compact objects in their product is the product of their respective subcategories of compact objects, i.e.,*

$$\left(\prod_{i\leq n} \mathcal{C}_i\right)^\omega \simeq \prod_{i\leq n} (\mathcal{C}_i)^\omega.$$

Proof. Let $\mathcal{D} := \prod_{i\leq n} \mathcal{C}_i$, and for all $i \leq n$, let $\pi_i : \mathcal{D} \to \mathcal{C}_i$ be the projection. Let $d \in \mathcal{D}$. We want to show that d is compact in $\mathcal{D}$ if and only if $\pi_i(d)$ is compact in $\mathcal{C}_i$ for all $i \leq n$. Assume first $d \in \mathcal{D}^\omega$. Let $i \leq n$, without loss of generality $i = 1$, and $\mathsf{colim}_{j\in\mathcal{J}} c_j$ be a filtered colimit diagram in $\mathcal{C}_1$. Note that the filtered colimit of a constant diagram on a certain object is always this object again (it holds by [Lur09,

Prop. 4.3.1.12] for colimits indexed weakly contractible categories, in particular for filtered categories by [Lur09, Lem. 5.3.1.18]). By assumption, we have equivalences

$$\begin{aligned}\mathsf{Map}_{\mathcal{C}_1}(\pi_1(d), \mathsf{colim}_{j\in\mathcal{J}} c_j) \times \prod_{\ell>1} \mathsf{Map}_{\mathcal{C}_\ell}(\pi_\ell(d),\, \mathsf{colim}_{j\in\mathcal{J}}\pi_\ell(d)) & \\ \simeq \mathsf{Map}_{\mathcal{D}}(d, \mathsf{colim}_{j\in\mathcal{J}}(c_j, \pi_2(d), \dots, \pi_n(d))) & \\ \simeq \mathsf{colim}_{j\in\mathcal{J}} \mathsf{Map}_{\mathcal{D}}(d, (c_j, \pi_2(d), \dots, \pi_n(d))) & \\ \simeq \mathsf{colim}_{j\in\mathcal{J}} \left(\mathsf{Map}_{\mathcal{C}_1}(\pi_1(d), c_j) \times \prod_{\ell>1} \mathsf{Map}_{\mathcal{C}_\ell}(\pi_\ell(d), \pi_\ell(d)) \right). &\end{aligned}$$

Now, the natural map $\mathsf{colim}_{j\in\mathcal{J}} \mathsf{Map}_{\mathcal{C}_1}(\pi_1(d), c_j) \to \mathsf{Map}_{\mathcal{C}_1}(\pi_1(d), \mathsf{colim}_{j\in\mathcal{J}} c_j)$ is a retract of this equivalence (pick the identities in all $\mathsf{Map}_{\mathcal{C}_\ell}(\pi_\ell(d), \pi_\ell(d))$ on both sides), so it is an equivalence as well. The converse direction is [Lur09, Lem. 5.3.4.10(2)]. □

Proposition 2.11. *Let S be a qcqs scheme, and let $f : X \to Y$ be a morphism in Sm_S.*

(i) The full subcategory of compact objects $\mathsf{SH}(X)^\omega \subseteq \mathsf{SH}(X)$ is the thick subcategory generated by the family E_X from Theorem 2.8. It is a symmetric monoidal subcategory, and is essentially small.

(ii) The pullback $f^* : \mathsf{SH}(Y) \to \mathsf{SH}(X)$ from Theorem 2.7(ii) restricts to $f^* : \mathsf{SH}(Y)^\omega \to \mathsf{SH}(X)^\omega$.

(iii) If furthermore f is smooth, the map $f_\sharp : \mathsf{SH}(X) \to \mathsf{SH}(Y)$ from Theorem 2.7(iv) restricts to $f_\sharp : \mathsf{SH}(X)^\omega \to \mathsf{SH}(Y)^\omega$.

(iv) Let $A, B \in \mathsf{Sm}_S$. The equivalence $\mathsf{SH}(A \amalg B) \simeq \mathsf{SH}(A) \times \mathsf{SH}(B)$ from Theorem 2.7(i) restricts to an equivalence $\mathsf{SH}(A \amalg B)^\omega \simeq \mathsf{SH}(A)^\omega \times \mathsf{SH}(B)^\omega$.

Proof. The first part of (i) follows from the fact that, by Lemma 2.9(i) and Theorem 2.8, $\mathsf{SH}(X)^\omega$ is the thick subcategory generated by the essentially small family E_X. To prove that $\mathsf{SH}(X)^\omega \subseteq \mathsf{SH}(X)$ is a symmetric monoidal subcategory, by [Lur09, Lem. 5.4.2.4] it suffices to show that it is closed under the tensor product and contains the unit. For the latter, note that the unit even belongs to E_X. The former follows from the description $\mathsf{SH}(X)^\omega = \mathsf{thick}(E_X)$, the fact that E_X is closed under the tensor product, and the fact that the tensor product preserves colimits in both variables. This proves part (i).

To prove (ii), by Lemma 2.9, it suffices to show that f^* maps the compact generators of Theorem 2.8 for $\mathsf{SH}(Y)$ to compact objects in $\mathsf{SH}(X)$. For any $U \in \mathsf{Sm}_Y$, by Theorem 2.7(ii)-(iii) we have $f^*((\mathbb{P}^1_Y)^{\wedge -n} \wedge \Sigma^\infty_+ U) \simeq (\mathbb{P}^1_X)^{\wedge -n} \wedge \Sigma^\infty_+ f^*(U) \in E \subseteq \mathsf{SH}(X)^\omega$, as needed.

Part (iii) is proven similarly: for any $U \in \mathsf{Sm}_X$, by Theorem 2.7(iv)-(v), we have

$$f_\sharp((\mathbb{P}^1_X)^{\wedge -n} \wedge \Sigma^\infty_+ U) \simeq f_\sharp((f^*\mathbb{P}^1_Y)^{\wedge -n} \wedge \Sigma^\infty_+ U) \simeq (\mathbb{P}^1_Y)^{\wedge -n} \wedge \Sigma^\infty_+ f_\sharp U.$$

Finally, part (iv) is a particular case of Lemma 2.10. □

2.2.2. *Construction of the motivic Thom spectrum functor.*

As mentioned before, we want to view SH^ω as a spherical presheaf of essentially small symmetric monoidal categories, inducing a spherical presheaf of spaces $\mathsf{SH}^{\omega,\simeq} : \mathsf{Sm}_S^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Spc}^\times)$.

Definition 2.12 ([Lur09, Def. 5.5.8.8])**.** Let $\mathcal{C}$ be an essentially small category with finite coproducts. Then, the category of *spherical presheaves (of spaces) on $\mathcal{C}$*, denoted by $\mathcal{P}_\Sigma(\mathcal{C})$, is the category of functors $\mathcal{C}^{\mathsf{op}} \to \mathsf{Spc}$ preserving finite products (i.e., sending finite coproducts in $\mathcal{C}$ to products in Spc). This construction defines a functor $\mathcal{P}_\Sigma(\bullet)$ from essentially small categories to sifted-cocomplete categories, which has a partial right adjoint given by the inclusion of essentially small sifted-cocomplete categories. In other terms, $\mathcal{P}_\Sigma(\mathcal{C})$ is the free sifted-cocomplete category generated by $\mathcal{C}$ [Lur09, Prop. 5.5.8.15], and every spherical presheaf is a sifted colimit of representable presheaves [Lur09, Lem. 5.5.8.14], which are in particular spherical.

The motivic Thom spectrum functor will be defined as a particular instance of the more general motivic colimit functors. Given $\mathcal{F}$ a spherical presheaf of essentially small symmetric monoidal categories on Sm_S, with good functoriality properties inspired by the ones of SH studied in the previous subsection, we will construct symmetric monoidal functors $\mathcal{P}_\Sigma(\mathsf{Sm}_{S'})_{/\mathcal{F}^\simeq} \to \mathcal{F}(S')$ for all $S' \in \mathsf{Sm}_S$. The construction of these motivic colimit functors is itself functorial in S', in the sense that we will view both the source and target categories as presheaves of (not necessarily small) symmetric monoidal categories on Sm_S, and construct a natural symmetric monoidal transformation between these presheaves.

Applying the construction to $\mathcal{F} = \mathsf{SH}^\omega$ produces the motivic Thom spectrum functor, but we will also apply the construction to another presheaf $\mathcal{R}$ related to real realization (Subsection 2.3.2) in order to compare the motivic Thom spectrum functor with the topological one (Subsection 2.3.3).

Before proceeding to the construction itself, we need another point of view on spherical presheaves of symmetric monoidal categories (on Sm_S). As we now explain, the latter may be viewed as functors $\mathsf{Span}(\mathsf{Sm}_S, \mathsf{all}, \mathsf{fold}) \to \mathsf{Cat}_\infty$ on a certain category of spans, that preserve finite products. Recall that the objects in Cat_∞ are *essentially small* categories.

Definition 2.13 ([Bar17, Section 5])**.** Let $\mathcal{C}$ be a category, and let left and right be two classes of edges in $\mathcal{C}$, both containing equivalences and closed under pullback along one another. Then, there is a category $\mathsf{Span}(\mathcal{C}, \mathsf{left}, \mathsf{right})$ with vertices the objects of $\mathcal{C}$, and for any $X, Z \in \mathcal{C}$, the edges from X to Z are given by all spans $X \leftarrow Y \to Z$ where $Y \in \mathcal{C}$, and $(Y \to X) \in \mathsf{left}$, $(Y \to Z) \in \mathsf{right}$. The composition of two spans $V \leftarrow W \to X$ and $X \leftarrow Y \to Z$ is the span $V \leftarrow W \times_X Y \to Z$.

Definition 2.14. Let Fin be the category of finite sets, and Fin_* be the category of finite pointed sets and pointed maps. For all $n \in \mathbb{N}$, let $\langle n \rangle := \{*, 1, \dots, n\} \in \mathsf{Fin}_*$, and $\langle n \rangle^\circ := \{1, \dots, n\} \in \mathsf{Fin}$.

Proposition 2.15 ([BH21, Prop. C.1])**.** *Let $\mathcal{D}$ be a category with finite products, and let* all *be the class of all edges in $\mathcal{D}$. Consider the functor $\theta : \mathsf{Fun}(\mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}), \mathcal{D}) \longrightarrow \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{D})$ induced by restriction along the functor $\mathsf{Fin}_* \to \mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all})$ sending $\langle n \rangle$ to $\langle n \rangle^\circ$, and a map $(f : \langle n \rangle \to \langle m \rangle)$ to $(\langle n \rangle^\circ \hookleftarrow f^{-1}(\langle m \rangle^\circ) \to \langle m \rangle^\circ)$. Then, θ restricts to an equivalence of categories*

$$\mathsf{Fun}^\times(\mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}), \mathcal{D}) \xrightarrow{\simeq} \mathsf{CAlg}(\mathcal{D}^\times).$$

This statement generalizes to presheaves with values in $\mathsf{CAlg}(\mathcal{D}^\times)$:

Proposition 2.16 ([BH21, Prop. C.5])**.** *For an extensive category $\mathcal{C}$ (e.g. Sm_S, see [BH21, Def. 2.3]), consider the functor*

$$\chi : \mathcal{C}^{\mathsf{op}} \times \mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}) \longrightarrow \mathsf{Span}(\mathcal{C}, \mathsf{all}, \mathsf{fold})$$
$$(c, \langle n \rangle^\circ) \longmapsto c^{\amalg n}$$
$$(c' \xrightarrow{f^{\mathsf{op}}} c, \langle n \rangle^\circ \xleftarrow{a} \langle m \rangle^\circ \xrightarrow{b} \langle k \rangle^\circ) \longmapsto (c'^{\amalg n} \leftarrow c^{\amalg m} \to c^{\amalg k})$$

where the restriction of $c^{\amalg m} \to c^{\amalg k}$ to the i-th component is the inclusion as the $b(i)$-th component, and the restriction of $c^{\amalg m} \to (c')^{\amalg}$ to the i-th component is given by f followed by the inclusion as the $a(i)$-th component. Then, for $\mathcal{D}$ a category with finite products, restriction along χ induces a functor $\Theta : \mathsf{Fun}(\mathsf{Span}(\mathcal{C}, \mathsf{all}, \mathsf{fold}), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathsf{Span}(\mathsf{Fin}, \mathsf{all}, \mathsf{all}), \mathcal{D})$, which restricts to an equivalence

$$\mathsf{Fun}^\times(\mathsf{Span}(\mathcal{C}, \mathsf{all}, \mathsf{fold}), \mathcal{D}) \xrightarrow{\simeq} \mathsf{Fun}^\times(\mathcal{C}^{\mathsf{op}}, \mathsf{CAlg}(\mathcal{D}^\times)).$$

Using this point of view, we are finally ready for the construction of our motivic colimit functors.

Notation 2.17. When working over a fixed base scheme S, denote $\mathsf{Span} := \mathsf{Span}(\mathsf{Sm}_S, \mathsf{all}, \mathsf{fold})$.

Theorem 2.18 ([BH21, §16.3])**.** *Let $\mathcal{F} : \mathsf{Span} \longrightarrow \mathsf{Cat}_\infty$ be a functor preserving finite products, such that:*

(i) for any $f : Y \to X$ in Sm_S smooth*, $f^* := \mathcal{F}(X \leftarrow Y = Y) : \mathcal{F}(X) \to \mathcal{F}(Y)$ admits a left adjoint, denoted by $f_\sharp$.*

(ii) for any Cartesian square in Sm_S

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f and f' smooth morphisms, the exchange transformation $\mathsf{Ex}^*_\sharp : f'_\sharp (g')^* \Longrightarrow g^* f_\sharp$ of functors $\mathcal{F}(X) \to \mathcal{F}(Y')$ (defined as in Theorem 2.7) is an equivalence.

(iii) Given $(\nabla : Y \to Z) \in \mathsf{fold}$, let $\nabla_\otimes := \mathcal{F}(Y = Y \to Z) : \mathcal{F}(Y) \to \mathcal{F}(Z)$. Then ∇ encodes the tensor product on the symmetric monoidal category $\mathcal{F}(Z)$ (Proposition 2.16). For every diagram in Sm_S of the form

$$\begin{array}{ccccc} W & \longleftarrow & R_{Y/X}(W \times_X Y) \times_Z Y & \xrightarrow{\nabla'} & R_{Y/X}(W \times_X Y) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u'} & \lrcorner & \downarrow{\scriptstyle u} \\ X & \xleftarrow{\quad f \quad} & Y & \xrightarrow{\qquad \nabla \qquad} & Z \end{array}$$

with u and u' smooth morphisms and $\nabla, \nabla' \in \mathsf{fold}$, the distributivity transformation

$$\mathsf{Dis}_{\sharp\otimes} : u_\sharp \nabla'_\otimes (f')^* \Longrightarrow \nabla_\otimes (\pi_Y)_\sharp \pi_W^*$$

of functors $\mathcal{F}(W) \to \mathcal{F}(Z)$ (defined as in Theorem 2.7) is an equivalence.

Then, there exists a natural transformation $M_{\mathcal{F}} : (\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to \mathcal{F}$ of finite-product-preserving functors $\mathsf{Span} \to \mathsf{Cat}_\infty$ (i.e., of spherical presheaves of symmetric monoidal categories).

Assume furthermore that $\mathcal{F}$ is a subfunctor of $\mathcal{G} : \mathsf{Span} \to \widehat{\mathsf{Cat}}_\infty$, a spherical presheaf of non-necessarily small symmetric monoidal categories. Then, if $\mathcal{G}$ lifts to the category of (not necessarily small) sifted-cocomplete categories and functors preserving sifted colimits, then we obtain a natural transformation $M_{\mathcal{F}} : \mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to \mathcal{G}$ of spherical of presheaves of (not necessarily small) sifted-cocomplete symmetric monoidal categories.

Remark 2.19. In the notation $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$, $\mathcal{F}^\simeq$ is viewed as a functor associating to $(u : X \to S) \in \mathsf{Sm}_S$ a presheaf of spaces on Sm_X, given by

$$\mathsf{Sm}_X^{\mathsf{op}} \xhookrightarrow{u_\sharp} \mathsf{Sm}_S^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Cat}_\infty \xrightarrow{(-)^\simeq} \mathsf{Spc}.$$

Thus, $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$ associates to X the slice category $\mathcal{P}_\Sigma(\mathsf{Sm}_X)_{/\mathcal{F}^\simeq}$, viewing $\mathcal{F}^\simeq \in \mathcal{P}_\Sigma(\mathsf{Sm}_X)$. It is a priori unclear whether the symmetric monoidal structure on the slice category induced by this construction agrees with the one in Proposition B.3. We will come back to this question in Subsection 2.3.3.

Notation 2.20. We denote the symmetric monoidal structure on $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\omega,\simeq}}$ induced by Theorem 2.18 by $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathsf{SH}^{\omega,\simeq}}$. By Proposition B.3, the underlying slice category admits another symmetric monoidal structure. We denote this second monoidal structure by $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})^{\otimes}_{/\mathsf{SH}^{\omega,\simeq}}$. A priori, we do not know if they agree; we will come back to this question right after the proof of Theorem 2.18.

Proof of Theorem 2.18. For completeness, we repeat the construction in [BH21, §16.3] in our more general context. By straightening-unstraightening [Lur09, §3.2], the functor $\mathcal{F}$ classifies a Cartesian fibration $F : \mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$. Let $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ be the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{Span})$ consisting of functors whose image is a span of the form $X \leftarrow Y = Y$ with $Y \to X$ a smooth morphism in Sm_S. The composition

$$\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span}) \times \Delta^1 \xrightarrow{\mathrm{ev}} \mathsf{Span} \xrightarrow{\mathcal{F}} \mathsf{Cat}_\infty$$

where the first functor is evaluation, can be viewed as a natural transformation between the functors $\mathcal{F} \circ s$ and $\mathcal{F} \circ t$, where s (resp. t) is the source (resp. target) functor $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span}) \to \mathsf{Fun}_{\mathsf{sm}}(\Delta^0, \mathsf{Span}) \simeq \mathsf{Span}$ induced by the inclusion $\{0\} \to \Delta^1$ (resp. $\{1\} \to \Delta^1$). Since precomposition of functors corresponds to pullbacks of Cartesian fibrations [Lur09, Def. 3.3.2.2], the functors $\mathcal{F} \circ s$ and $\mathcal{F} \circ t$ classify the Cartesian fibrations s^*F and t^*F, respectively, defined as the pullbacks

$$\begin{array}{ccc} s^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ {\scriptstyle s^*F}\downarrow & \lrcorner & \downarrow{\scriptstyle F} \\ \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} & \xrightarrow{s^{\mathsf{op}}} & \mathsf{Span}^{\mathsf{op}} \end{array} \qquad \begin{array}{ccc} t^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ {\scriptstyle t^*F}\downarrow & \lrcorner & \downarrow{\scriptstyle F} \\ \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} & \xrightarrow{t^{\mathsf{op}}} & \mathsf{Span}^{\mathsf{op}}, \end{array}$$

and the natural transformation $\mathcal{F} \circ s \Rightarrow \mathcal{F} \circ t$ gives a morphism of Cartesian fibrations $\phi : s^*\mathcal{E} \to t^*\mathcal{E}$ (i.e., a functor over $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}}$ that preserves Cartesian edges). We have a commutative diagram

$$\begin{array}{ccccc} t^*\mathcal{E} & \xleftarrow{\phi} & s^*\mathcal{E} & \xrightarrow{\chi} & \mathcal{E} \\ & {\scriptstyle t^*F}\searrow & {\scriptstyle s^*F}\downarrow & & \downarrow{\scriptstyle F} \\ & & \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} & \xrightarrow{s^{\mathsf{op}}} & \mathsf{Span}^{\mathsf{op}}. \end{array}$$

We now pause to give an outline of the rest of the proof. Our goal is to show that the composition $s^{\mathsf{op}} \circ t^*F : t^*\mathcal{E} \to \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}} \to \mathsf{Span}^{\mathsf{op}}$ is a Cartesian fibration, classified by a presheaf of symmetric monoidal categories $(\mathsf{Sm}_\bullet)_{/\!/\mathcal{F}}$, of which $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ is a subfunctor; and to construct a morphism of Cartesian fibrations $t^*\mathcal{E} \to \mathcal{E}$. The latter corresponds to the natural transformation $M_{\mathcal{F}}$ we are looking for. Here are the steps we will follow:

(1) Show that s^{op} is a Cartesian fibration, so that the composition $s^{\mathsf{op}} \circ t^*F : t^*\mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$ is a Cartesian fibration (note that t^*F is a Cartesian fibration because it is a pullback of F).
(2) Define a *relative left adjoint* ψ to ϕ, which will in particular be a map $t^*\mathcal{E} \to s^*\mathcal{E}$.
(3) Show that the composition $\chi \circ \psi$ is a morphism of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$, and thus corresponds to a natural transformation between the functors classifying these Cartesian fibrations.

(4) Show that $s^{\mathsf{op}} \circ t^* F$ is indeed classified by some presheaf $(\mathsf{Sm}_\bullet)_{/\!/\mathcal{F}}$ admitting $(\mathsf{Sm}_\bullet)_{/\mathcal{F}}$ as a subfunctor, and deduce the first part of the statement of the theorem.
(5) Prove the second part of the statement, i.e., the extension to sifted-cocompletions.

Step 1: *The functor s^{op} is a Cartesian fibration.* Equivalently, we show that s is a coCartesian fibration. Firstly, it is an inner fibration because the source functor $\mathsf{Fun}(\Delta^1, \mathsf{Span}) \to \mathsf{Span}$ is an inner fibration by [Lur09, Cor. 2.4.7.11], and the restriction of an inner fibration to a full subcategory is still an inner fibration by [Lur24, Tag 01CU]. Secondly, we claim that a coCartesian edge over $X \xleftarrow{f} Y \xrightarrow{\nabla} Z$ in Span with source $X \xleftarrow{j} W = W$ is given by the edge in $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ (i.e., natural transformation)

$$\begin{array}{ccccc}
W & \longleftarrow & R_{Y/Z}(W \times_X Y) \times_Z Y & \longrightarrow & R_{Y/Z}(W \times_X Y) \\
\| & & \| & & \| \\
W & \xleftarrow{f'} & R_{Y/Z}(W \times_X Y) \times_Z Y & \xrightarrow{\nabla'} & R_{Y/Z}(W \times_X Y) \\
{\scriptstyle j}\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{f} & Y & \xrightarrow{\nabla} & Z,
\end{array}$$

which we call e (for a grid like this to define a natural transformation between the vertical spans on the left and right-hand sides, we need the top left and bottom right squares to be pullbacks, which is the case here). Indeed, given another edge e' with the same source

$$\begin{array}{ccccc}
W & \longleftarrow & B & \longrightarrow & D \\
\| & & \| & & \| \\
W & \xleftarrow{g'} & B & \xrightarrow{\widetilde{\nabla}'} & D \\
{\scriptstyle j}\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{g} & A & \xrightarrow{\widetilde{\nabla}} & C
\end{array}$$

and a commutative diagram in Span

$$\begin{array}{ccccc}
X & \xleftarrow{f} & Y & \xrightarrow{\nabla} & Z \\
& {\scriptstyle g}\nwarrow & & & \uparrow{\scriptstyle h} \\
& & A & & E \\
& & & {\scriptstyle \widetilde{\nabla}}\searrow & \downarrow{\scriptstyle \overline{\nabla}} \\
& & & & C
\end{array}$$

(in particular $A \simeq Y \times_Z E$), the following edge e'' in $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ lifts $Z \leftarrow E \to C$ and provides a commutative diagram $e'' \circ e \simeq e'$

$$\begin{array}{ccccc}
R_{Y/Z}(W \times_X Y) & \longleftarrow & D \times_C E & \longrightarrow & D \\
\| & & \| & & \| \\
R_{Y/Z}(W \times_X Y) & \xleftarrow{h'} & D \times_C E & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow \\
Z & \xleftarrow{h} & E & \xrightarrow{\overline{\nabla}} & C.
\end{array}$$

To construct h', note that by definition of the Weil restriction, there are equivalences

$$\begin{aligned}
\mathsf{Sch}_Z(D \times_C E, R_{Y/Z}(W \times_X Y)) &\simeq \mathsf{Sch}_Y((D \times_C E) \times_Z Y, W \times_X Y) \\
&\simeq \mathsf{Sch}_Y(D \times_C A, W \times_X Y) \\
&\simeq \mathsf{Sch}_Y(B, W \times_X Y).
\end{aligned}$$

Then, under these equivalences, h' corresponds to the map in $\mathsf{Sch}_Y(B, W \times_X Y)$ induced by $g' : B \to W$. The bottom right square in e'' is determined by the requirement of it being a pullback, and the map h' is determined, under the equivalences we just saw, by the condition $e'' \circ e = e'$, so that the lift e'' is essentially unique.

Step 2: *The functor ϕ has a relative left adjoint.* This means that there exists a functor ψ in the other direction, and a transformation $\varepsilon : \psi\phi \to \mathsf{id}$ exhibiting ψ as a left adjoint to ϕ, and moreover ψ

commutes with the structure maps, in the sense that $s^*F \circ \psi \simeq t^*F \circ \phi \circ \psi \Rightarrow t^*F$ is an equivalence [Lur17, Def. 7.3.2.2]. By [Lur17, Prop. 7.3.2.6], since ϕ is a morphism of Cartesian fibrations, it suffices to show that ϕ has fiberwise a left-adjoint. Over $(X \leftarrow Y = Y) \in \mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ (so $f : Y \to X$ is smooth), the functor ϕ is by definition the corresponding component of the natural transformation $\mathcal{F} \circ s \to \mathcal{F} \circ T$ it encodes. Therefore, it is given by $f^* : \mathcal{F}(X) \to \mathcal{F}(Y)$, which admits a left adjoint $f_\sharp$ by assumption (i). We thus obtain a relative left adjoint $\psi : t^*\mathcal{E} \to s^*\mathcal{E}$.

Step 3: *The composition $\chi \circ \psi : t^*\mathcal{E} \to s^*\mathcal{E} \to \mathcal{E}$ is a morphism of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$.* Compatibility with the structure maps down to $\mathsf{Span}^{\mathsf{op}}$ holds by construction. So we have to show that $\chi \circ \psi$ preserves Cartesian edges. By [Lur24, Tag 01UL], for Cartesian fibrations $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{C} \to \mathcal{D}$, an edge e in $\mathcal{E}$ is $(q \circ p)$-Cartesian if and only if e is p-Cartesian and $p(e)$ is q-Cartesian. In our situation, if e is an $(s^{\mathsf{op}} \circ t^*F)$-Cartesian edge, e is (t^*F)-Cartesian and $(t^*F)(e)$ is s^{op}-Cartesian. It follows that $(t^*F)(e)$ is the opposite of an edge in $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})$ of the form

$$\begin{array}{ccccc} W & \longleftarrow & Y' & \longrightarrow & Z' \\ \| & & \| & & \| \\ W & \xleftarrow{f'} & Y' & \xrightarrow{\nabla'} & Z' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u'} & \lrcorner & \downarrow{\scriptstyle u} \\ X & \xleftarrow[f]{} & Y & \xrightarrow[\nabla]{} & Z \end{array} \tag{1}$$

where $Y' = R_{Y/Z}(W \times_X Y) \times_Z Y$ and $Z' = R_{Y/Z}(W \times_X Y)$. Here g, u, and u' are smooth. Since e is a (t^*F)-Cartesian lift of this edge, it is of the form $\alpha : (Z \leftarrow Z' = Z', H) \to (X \leftarrow W = W, E)$ with $H \in \mathcal{F}(Z')$, $E \in \mathcal{F}(W)$, and α consists in the data of $(t^*F)(e)$ (the diagram above) together with an equivalence $H \xrightarrow{\simeq} \nabla'_\otimes f'^* E$. Then, $\chi\psi(e)$ is an edge $(Z, u_\sharp(H)) \to (X, g_\sharp(E))$ given by the morphisms $(X \leftarrow Y \to Z)^{\mathsf{op}}$ and $u_\sharp H \to \nabla_\otimes f^* g_\sharp E$, where the latter is the composite

$$u_\sharp H \longrightarrow u_\sharp \nabla'_\otimes f'^* E \xrightarrow{\mathsf{Dis}_{\sharp,\otimes}} \nabla_\otimes (\pi_Y)_\sharp \pi_W^* E \xrightarrow{\mathsf{Ex}^*_\sharp} \nabla_\otimes f^* g_\sharp E,$$

with $\pi_Y : Y \times_X W \to Y$ and $\pi_W : Y \times_X W \to W$ the projections. We saw above that the first map in the composition was an equivalence, and the two other ones are equivalences by assumptions (iii) and (ii) in the statement respectively. Therefore, $\chi\psi(e)$ is F-Cartesian, as desired.

Step 4: *We deduce the first part of the statement.* The morphism of Cartesian fibrations $\chi\psi$ corresponds to a natural transformation $M_{\mathcal{F}}$ of functors $\mathsf{Span} \to \mathsf{Cat}_\infty$ between the functors classified by $s^{\mathsf{op}} \circ t^*F$ and F, respectively. The former associates to $X \in \mathsf{Sm}_S$ the fiber of $s^{\mathsf{op}} \circ t^*F$ over X. The fiber of s^{op} over X can be identified with Sm_X; and over an object Y in Sm_X, the fiber of t^*F is by definition $\mathcal{F}(Y)$. Therefore, the functor classified by $s^{\mathsf{op}} \circ t^*F$ may be denoted by $(\mathsf{Sm}_\bullet)_{/\!/\mathcal{F}}$: it associates to $X \in \mathsf{Sm}_S$ the category $(\mathsf{Sm}_X)_{/\!/\mathcal{F}}$ of pairs (U, E) with $U \in \mathsf{Sm}_X$ and $E \in \mathcal{F}(U)$, and a morphism $(U, E) \to (U', E')$ is the data of $f : U \to U'$ in Sm_X and a map $E \to \mathcal{F}(f^{\mathsf{op}})(E')$ in $\mathcal{F}(U)$ (*not* required to be an equivalence). These categories splice into a presheaf as follows. Given a morphism $X \xleftarrow{f} Y \xrightarrow{\nabla} Z$ in Span, consider the Cartesian lift built in Step 1. Then $X \leftarrow Y \to Z$ is mapped to the functor $(\mathsf{Sm}_X)_{/\!/\mathcal{F}} \to (\mathsf{Sm}_Z)_{/\!/\mathcal{F}}$ sending a pair (W, E) with $W \in \mathsf{Sm}_X$ and $E \in \mathcal{F}(X')$ to the pair consisting of $R_{Y/Z}(X' \times_X Y) \in \mathsf{Sm}_Z$ and $\nabla'_\otimes f'^* E \in \mathcal{F}(R_{Y/Z}(X' \times_X Y))$. The slice category $(\mathsf{Sm}_X)_{/\mathcal{F}^\simeq}$ is viewed as the wide subcategory of $(\mathsf{Sm}_X)_{/\!/\mathcal{F}}$ where the maps $E \to \mathcal{F}(f^{\mathsf{op}})(E')$ in $\mathcal{F}(U)$ in the description above are required to be equivalences. Then, $(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$ forms a subpresheaf of $(\mathsf{Sm}_\bullet)_{/\!/\mathcal{F}}$, to which the natural transformation $M_{\mathcal{F}}$ restricts. This proves the first part of the statement.

Step 5: *We prove the second part of the statement.* If $\mathcal{F}$ embeds into a presheaf of not necessarily small symmetric monoidal categories $\mathcal{G}$ that lifts (as a functor with source category Span) to sifted-cocomplete categories, we have a composite morphism of presheaves (which can also be viewed as a natural transformation of functors on Span) $(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to \mathcal{F} \to \mathcal{G}$. Since $\mathcal{G}$ lifts to sifted-cocomplete categories, by the universal property of the sifted-cocompletion (see Definition 2.12), this morphism left Kan extends to a transformation $\mathcal{P}_\Sigma((\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}) \simeq \mathcal{P}_\Sigma((\mathsf{Sm}_\bullet))_{/\mathcal{F}^\simeq} \to \mathcal{F} \to \mathcal{G}$ (the equivalence on the left hand-side is Proposition B.2), where $\mathcal{P}_\Sigma((\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq})$ corresponds to the composition

$$\mathsf{Span} \xrightarrow{(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}} \mathsf{Cat}_\infty \xrightarrow{\mathcal{P}_\Sigma(\bullet)} \widehat{\mathsf{Cat}}^{\mathsf{sift}}_\infty.$$

This is the desired morphism of presheaves of symmetric monoidal categories $M_{\mathcal{F}} : \mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to \mathcal{G}$. □

Definition 2.21. Let $\mathcal{F}$ be a presheaf on Sm_S as in Theorem 2.18. The *motivic colimit functor associated with* $\mathcal{F}$ is the S-component of the transformation $M_{\mathcal{F}}$. Abusing notations, we also denote it by $M_{\mathcal{F}}$.

Notation 2.20 gives rise to the following question.

Question 2.22. *In the situation of Theorem 2.18, do the symmetric monoidal structures $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes'}_{/\mathcal{F}}$ (obtained from the construction of Theorem 2.18, i.e., from [BH21, §16.3]) and $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{/\mathcal{F}}$ (described in Proposition B.3) agree?*

We conjecture that this question can be answered by the affirmative, but we were unable to prove it. We prove a weaker result in Theorem 2.44. Since both symmetric monoidal structures are by construction Day convolution with respect to some symmetric monoidal structures on the slice $(\mathsf{Sm}_S)_{/\mathcal{F}}$, it would suffice to prove that the structures $\otimes$ and $\otimes'$ agree at this level. Here is a hint in the direction of a positive answer to Question 2.22:

Proposition 2.23. *In the situation of Theorem 2.18, the tensor product of any two objects for the symmetric monoidal structures $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes'}_{/\mathcal{F}}$ and $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{/\mathcal{F}}$ agree.*

Proof. Let $g : X \to \mathcal{F}$ and $h : Y \to \mathcal{F}$ be two objects in $(\mathsf{Sm}_S)_{/\mathcal{F}}$, represented by $x \in \mathcal{F}(X)$ and $y \in \mathcal{F}(Y)$, respectively. By the informal description of the tensor product in $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{/\mathcal{F}}$ in Proposition B.3, we have

$$(X \to \mathcal{F}) \otimes (Y \to \mathcal{F}) = X \times Y \xrightarrow{g\times h} \mathcal{F} \times \mathcal{F} \xrightarrow{\mu} \mathcal{F}$$

where μ gives the multiplicative structure on the presheaf of symmetric monoidal categories $\mathcal{F}$. The resulting map $X \times Y \longrightarrow \mathcal{F}$ is represented by the element $\mathcal{F}(\pi_X^{\mathsf{op}})(x) \otimes_{X\times Y} \mathcal{F}(\pi_Y^{\mathsf{op}})(y) \in \mathcal{F}(X \times Y)$, where for $Z \in \mathsf{Sm}_{\mathbb{R}}$, the functor $\otimes_Z$ is the tensor product on $\mathcal{F}(Z)$, namely $\mathcal{F}(Z^{\amalg 2} \rightrightarrows Z^{\amalg 2} \longrightarrow Z)$ (viewing $\mathcal{F}$ as a functor $\mathsf{Span} \to \mathsf{Cat}_\infty$ preserving finite products).

On the other hand, recall that the presheaf of symmetric monoidal categories $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)^{\otimes'}_{/\mathcal{F}}$ is classified by the Cartesian fibration $s^{\mathsf{op}} \circ t^*F : t^*\mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$ (in the notation of the proof of Theorem 2.18). Thus, the tensor product of g and h is the source of an $(s^{\mathsf{op}} \circ t^*F)$-Cartesian lift for $(S^{\amalg 2} = S^{\amalg 2} \to S)^{\mathsf{op}}$ with target $(X \amalg Y, (x, y) \in \mathcal{F}(X \amalg Y) \simeq \mathcal{F}(X) \times \mathcal{F}(Y)) \in t^*\mathcal{E}$. We have seen in the proof of Theorem 2.18 that an s^{op}-Cartesian lift for such an edge was given by (the opposite of)

$$\begin{array}{ccccc} X \amalg Y & \longleftarrow & (X \times Y)^{\amalg 2} & \xrightarrow{\nabla'} & X \times Y = R_{S^{\amalg 2}/S}(X \amalg Y) \\ \downarrow & & \downarrow & & \downarrow \\ S^{\amalg 2} & = & S^{\amalg 2} & \xrightarrow{\qquad\nabla\qquad} & S \end{array}$$

and thus the source of the $(s^{\mathsf{op}} \circ t^*F)$-Cartesian lift we are looking for is

$$\big(X \times Y, \mathcal{F}(X \amalg Y \longleftarrow (X \times Y)^{\amalg 2} \longrightarrow X \times Y)(x, y)\big),$$

which is exactly $\mathcal{F}(\pi_X^{\mathsf{op}})(x) \otimes_{X\times Y} \mathcal{F}(\pi_Y^{\mathsf{op}})(y) \in \mathcal{F}(X \times Y)$ by construction. □

Proposition 2.24. *Let $\mathcal{F}$ be as in Theorem 2.18. Then the associated motivic colimit functor $M_{\mathcal{F}}$ is a symmetric monoidal, colimit-preserving functor sending the arrow $u : y(X) \to \mathcal{F}^{\simeq}$ (where $X \in \mathsf{Sm}_S$ has structure map $f : X \to S$) to $f_\sharp(E)$, where $E = u(X)(\mathsf{id}_X) \in \mathcal{F}^{\simeq}(X)$ is classified by u. More generally, the image of an arrow $\gamma : \mathcal{G} \to \mathcal{F}^{\simeq}$ in $\mathcal{P}(\mathsf{Sm}_S)_{/\mathcal{F}^{\simeq}}$ is given by*

$$M_{\mathcal{F}} \simeq \mathsf{colim}_{(x,X)\in(\mathsf{Sm}_S)_{/\!/\mathcal{G}}} (p_X)_\sharp \gamma(x).$$

Proof. The last assertion is [BH21, Rmk. 16.5] or [BEH22, Rmk. 2.6], and is deduced from the expression of an element of the slice as a colimit of representable objects, and the fact that $M_{\mathcal{F}}$ preserves colimits (or viewing $M_{\mathcal{F}}$ as a left Kan extension). □

To obtain the motivic Thom spectrum functor, we now apply Definition 2.21 to the functor SH^ω.

Proposition 2.25. *The functor $\mathsf{SH}^\omega : \mathsf{Sm}_{\mathbb{R}}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Cat}_\infty^\times)$ satisfies all assumptions of Theorem 2.18; in the statement of the latter, we may choose $\mathcal{F} = \mathsf{SH}^\omega$ and $\mathcal{G} = \mathsf{SH}$. In particular, there is a motivic colimit functor $(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\omega,\simeq}} \to \mathsf{SH}^\omega(\mathbb{R})$, extending to a symmetric monoidal functor*

$$M := M_{\mathsf{SH}^{\omega,\simeq}} : \mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})_{/\mathsf{SH}^{\omega,\simeq}} \longrightarrow \mathsf{SH}(\mathbb{R}),$$

which we call the motivic Thom spectrum functor.

Proof of Proposition 2.25. By Theorem 2.7, the functor SH defines a spherical presheaf of symmetric monoidal categories that satisfies assumptions (i) to (iii) in Theorem 2.18. Then, by Proposition 2.11, SH^ω also satisfies these assumptions. Therefore, the statement follows as soon as we show that SH, viewed as a functor with source category Span with values in categories, lifts to sifted-cocomplete categories. For all $X \in \mathsf{Sm}_{\mathbb{R}}$, $\mathsf{SH}(X)$ is cocomplete, in particular sifted-cocomplete. Moreover, if $(X \xleftarrow{f} Y \xrightarrow{\nabla} Z)$ is a morphism in Span, then its image $\nabla_\otimes \circ f^* : \mathsf{SH}(X) \to \mathsf{SH}(Z)$ by SH preserves sifted colimits. Indeed, f^* preserves all colimits as a left adjoint, and ∇ is induced by the tensor product functor, which preserves sifted colimits since it preserves colimits in both variables separately. More precisely, if $\nabla : Z \amalg Z \to Z$ is the simplest fold map, then $\nabla_\otimes : \mathsf{SH}(Z) \times \mathsf{SH}(Z) \to \mathsf{SH}(Z)$ is the tensor product (Theorem 2.18(iii)). In this situation, for any sifted diagram $\mathcal{G} : \mathcal{D} \to \mathsf{SH}(Z) \times \mathsf{SH}(Z)$, we have

$$\begin{aligned}
\nabla_\otimes(\mathsf{colim}_{\mathcal{D}}\ \mathcal{G}) &\simeq \nabla_\otimes(\mathsf{colim}_{\mathcal{D}}\ \pi_1\mathcal{G}, \mathsf{colim}_{\mathcal{D}}\ \pi_2\mathcal{G}) \\
&\simeq (\mathsf{colim}_{\mathcal{D}}\ \pi_1\mathcal{G}) \otimes (\mathsf{colim}_{\mathcal{D}}\ \pi_2\mathcal{G}) \\
&\simeq \mathsf{colim}_{\mathcal{D}\times\mathcal{D}}\ (\pi_1\mathcal{G}\pi_1 \otimes \pi_2\mathcal{G}\pi_2) && (\otimes \text{ preserves colimits in both variables separately}) \\
&\simeq \mathsf{colim}_{\mathcal{D}}\ (\pi_1\mathcal{G} \otimes \pi_2\mathcal{G}) && (\text{by definition of a sifted category}) \\
&\simeq \mathsf{colim}_{\mathcal{D}}\ \nabla_\otimes\mathcal{G}
\end{aligned}$$

where π_1 and π_2 are the canonical projections for the products $\mathsf{SH}(Z) \times \mathsf{SH}(Z)$ and $\mathcal{D} \times \mathcal{D}$. The same holds for more general fold maps. This finishes the proof. □

Remark 2.26. By Proposition 2.24, the motivic Thom spectrum functor is informally given by the formula

$$M(\mathcal{G} \to \mathsf{SH}^\simeq) \simeq \mathsf{colim}_{(x,X)\in(\mathsf{Sm}_{\mathbb{R}})_{/\!/\mathcal{G}}}(p_X)_\sharp\gamma(x),$$

where p_X is the structure map $X \to \mathbb{R}$, and the colimit is taken in $\mathsf{SH}(\mathbb{R})$.

Example 2.27 ([BH21, §16.2 and Ex. 16.22], [BH20, below Lemma 4.6]). Paralleling Example 2.5, we now explain how to define $\mathcal{E}_\infty$-structures on the motivic cobordism spectra using the motivic Thom spectrum functor. The motivic $\mathcal{E}_\infty$-ring spectrum MGL is the image by M of the motivic j-homomorphism $j : K^\circ \longrightarrow \mathsf{SH}^{\omega,\simeq}$, where K° is the rank 0 summand of algebraic K-theory. As we will now see, j is constructed as a morphism of presheaves of symmetric monoidal categories, and thus acquires by Proposition 2.28 below the structure of a commutative algebra object in the source category of the functor M. For any $X \in \mathsf{Sm}_{\mathbb{R}}$, there is a symmetric monoidal category $\mathsf{Vect}(X)$ of algebraic vector bundles on X, where the tensor product is the direct sum of vector bundles. Pullback of vector bundles makes the construction contravariantly functorial in X. There is a symmetric monoidal functor $\mathsf{Vect}(X) \to \mathsf{Pic}(\mathsf{SH}(X))$ sending a vector bundle ξ to $\Sigma^\infty(\xi/(\xi \setminus \{0\}))$, where $\mathsf{Pic}(\mathsf{SH}(X))$ is the subgroupoid of $\mathsf{SH}(X)^\simeq$ spanned by the invertible objects. It extends to a morphism of presheaves of $\mathcal{E}_\infty$-spaces $\mathsf{Vect}^\simeq \to \mathsf{Pic}(\mathsf{SH}(-))$, which factors through the group completions $\mathsf{Vect}(X)^{\simeq,\mathsf{gp}}$ because $\mathsf{Pic}(\mathsf{SH}(X))$ is already group-like by construction. Thus, we obtain a morphism $K \to \mathsf{Pic}(\mathsf{SH}(-)) \to \mathsf{SH}^{\omega,\simeq}$ of spherical presheaves of group-like $\mathcal{E}_\infty$-spaces. Furthermore, the rank map $\mathsf{Vect}(X)^\simeq \to \mathbb{N}$ is a morphism of monoids, and taking group completions yields a morphism $K \to \mathbb{Z}$ of spherical presheaves of group-like $\mathcal{E}_\infty$-spaces, whose fiber is the rank zero part K°. Restricting our morphism $K \to \mathsf{SH}^{\omega,\simeq}$ to K°, we obtain the motivic j-homomorphism $j : K^\circ \longrightarrow \mathsf{SH}^{\omega,\simeq}$.

Similarly, the motivic spectra MSL and MSp are respectively given by $M(KSL^\circ \to K^\circ \xrightarrow{j} \mathsf{SH}^{\omega,\simeq})$ and $M(KSp^\circ \to K^\circ \xrightarrow{j} \mathsf{SH}^{\omega,\simeq})$, where KSL° and KSp° are defined similarly as K°, but with respect to even-dimensional oriented bundles and even-dimensional symplectic bundles, respectively. They can be described by $K^\circ \simeq \mathsf{L}_{\mathsf{mot}}BGL \simeq \mathsf{L}_{\mathsf{mot}}\mathsf{colim}_{n\in\mathbb{N}}BGL_n$, $KSL^\circ \simeq \mathsf{L}_{\mathsf{mot}}BSL$, and $KSp^\circ \simeq \mathsf{L}_{\mathsf{mot}}BSp$, respectively ([BH20, after Lem. 4.6] and [BH21, before Thm. 16.13]).

Proposition 2.28. *There is a functor*

$$\mathsf{Fun}^\times(\mathsf{Span}(\mathsf{Sm}_S, \mathsf{all}, \mathsf{fold}), \mathsf{Spc})_{/\mathsf{SH}^\omega} \longrightarrow \mathsf{CAlg}\left(\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes'}_{/\mathsf{SH}^{\omega,\simeq}}\right),$$

where $\mathsf{Fun}^\times$ *denotes the category of functors preserving finite products (and compatible natural transformations). In particular, a morphism of spherical presheaves* $\mathcal{E}_\infty$*-spaces* $A \to \mathsf{SH}^{\omega,\simeq}$ *defines a commutative algebra object in the slice* $\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes'}_{/\mathsf{SH}^{\omega,\simeq}}$.

Proof. This follows from the last displayed composition of functors in [BH21, proof of Prop. 16.17 and Rmk. 16.18]. □

Remark 2.29. This is a partial analog to Proposition B.5. Indeed, by Proposition B.5, a morphism of commutative algebras in $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$ with target $\mathsf{SH}^{\omega,\simeq}$ yields a commutative algebra in the slice

$\mathcal{P}_\Sigma(\mathsf{Sm}_S)^{\otimes}_{/\mathsf{SH}^{\omega,\simeq}}$. We don't know if the $\otimes$ and $\otimes'$ structures agree, but Proposition 2.28 tells us that, at least, we may construct some commutative algebras with respect to both structures in the same way.

2.3. **Comparison of the topological and motivic Thom spectrum functors.**

In this subsection, we aim to show that the motivic Thom spectrum functor corresponds under real realization to the topological Thom spectrum functor, in the appropriate sense. To do so, we will consider another motivic colimit functor that already incorporates real realization in its definition (Subsection 2.3.2). In order to compare the latter with the motivic Thom spectrum functor (Subsection 2.3.3), we first study in Subsection 2.3.1 naturality in $\mathcal{F}$ of the construction from Definition 2.21. Finally, in Subsection 2.3.4, we use this to compute the real realizations of MSL, MGL and MSp.

2.3.1. *Naturality of the motivic Thom spectrum functor in the presheaf $\mathcal{F}$.*

Proposition 2.30. *Let $\mathcal{F},\mathcal{G} : \mathsf{Span} \to \mathsf{Cat}_\infty$ be functors satisfying the assumptions of Theorem 2.18. Assume that $\tau : \mathcal{F} \to \mathcal{G}$ is a natural transformation such that, in the notation of assumption (i) in Theorem 2.18, for any smooth morphism $f : Y \to X$ in $\mathsf{Sm}_\mathbb{R}$, the canonical transformation $f_\sharp \tau_Y \Rightarrow \tau_X f_\sharp$ is an equivalence (see Remark 2.31 below). Then, postcomposition with τ induces a natural transformation $\tau_\sharp : (\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to (\mathsf{Sm}_\bullet)_{/\mathcal{G}^\simeq}$, fitting into a commutative diagram of transformations of functors $\mathsf{Span} \to \mathsf{Cat}_\infty$ preserving finite products:*

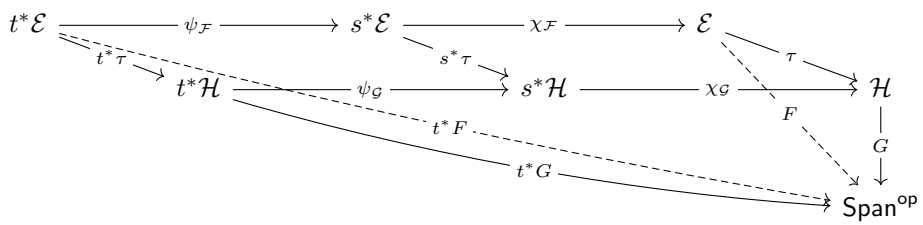
$$\begin{array}{ccc} (\mathsf{Sm}_\bullet)^{\simeq}_{/\mathcal{F}} & \xrightarrow{\tau_\sharp} & (\mathsf{Sm}_\bullet)^{\simeq}_{/\mathcal{G}} \\ {\scriptstyle M_\mathcal{F}}\downarrow & & \downarrow{\scriptstyle M_\mathcal{G}} \\ \mathcal{F} & \xrightarrow[\tau]{} & \mathcal{G}. \end{array} \tag{2}$$

Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are subpresheaves of some spherical presheaves of symmetric monoidal categories that lift to sifted-cocomplete categories, denoted by $\mathcal{F}'$ and $\mathcal{G}'$ respectively, and τ extends to a transformation $\mathcal{F}' \Rightarrow \mathcal{G}'$ of functors $\mathsf{Span} \to \widehat{\mathsf{Cat}}^{\mathsf{sift}}_\infty$, then the statement holds for $M_\mathcal{F}$ and $M_\mathcal{G}$ replaced with $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to \mathcal{F}'$ and $\mathcal{P}_\Sigma(\mathsf{Sm}_\bullet)_{/\mathcal{G}^\simeq} \to \mathcal{G}'$, respectively.

Remark 2.31. In the situation of Proposition 2.30, $\mathcal{F}$ and $\mathcal{G}$ are in particular Sm-*premotivic categories over* $\mathsf{Sm}_\mathbb{R}$ [EK20, §2.2]. That is, they are functors $\mathsf{Sm}_\mathbb{R}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ such that for each $(f : Y \to X) \in \mathsf{Sm}$ a smooth morphism in $\mathsf{Sm}_\mathbb{R}$, $f^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ has a left adjoint $f_\sharp$ (respectively, for $\mathcal{G}$). If each component of τ admits a left adjoint, our assumption on τ gives exactly the notion of a *premotivic adjunction*. Notice further that a transformation τ as in Proposition 2.30 automatically also preserves the units and counits of the adjunctions $f_\sharp \dashv f^*$ for any smooth morphism f. Indeed, applying τ to the unit and counit for an adjunction $f_\sharp : \mathcal{F}(Y) \leftrightarrows \mathcal{F}(X) : f^*$ yields transformations which exhibit $f_\sharp : \mathcal{G}(Y) \leftrightarrows \mathcal{G}(X) : f^*$ as an adjunction; they must therefore be equivalent to the unit and counit we were originally given for this adjunction [Lur24, Tag 02F4]. This holds very generally and may also be seen by viewing τ_Y as a morphism between the biCartesian fibrations over Δ^1 representing these adjunctions [Lur09, §5.2.2].

Proof of Proposition 2.30. Firstly, to construct $\tau_\sharp$, we have to produce a morphism between the Cartesian fibrations classified by $(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq}$ and $(\mathsf{Sm}_\bullet)_{/\mathcal{G}^\simeq}$. We use the notation of the proof of Theorem 2.18. The transformation τ induces a morphism between the Cartesian fibrations $F : \mathcal{E} \to \mathsf{Span}^{\mathsf{op}}$ and $G : \mathcal{H} \to \mathsf{Span}^{\mathsf{op}}$ classified by $\mathcal{F}$ and $\mathcal{G}$, respectively. It pulls back to morphisms of Cartesian fibrations $s^*\tau : s^*\mathcal{E} \to s^*\mathcal{H}$ and $t^*\tau : t^*\mathcal{E} \to t^*\mathcal{H}$ over $\mathsf{Fun}_{\mathsf{sm}}(\Delta^1, \mathsf{Span})^{\mathsf{op}}$. The latter defines a morphism of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$ between $s^{\mathsf{op}} \circ t^*F$ and $s^{\mathsf{op}} \circ t^*G$. Indeed, assume e is an $(s^{\mathsf{op}} \circ t^*F)$-Cartesian edge in $t^*\mathcal{E}$. Then, by [Lur24, Tag 01UL], e is t^*F-Cartesian and therefore $(t^*\tau)(e)$ is (t^*G)-Cartesian. By *loc. cit.*, $(t^*G)((t^*\tau)(e)) = (t^*F)(e)$ is s^{op}-Cartesian, and so $t^*\tau(e)$ is $(s^{\mathsf{op}} \circ t^*G)$-Cartesian.

Secondly, we show the commutativity of Diagram (2). We have a diagram



of morphisms of Cartesian fibrations over $\mathsf{Span}^{\mathsf{op}}$. We have to show that the top rectangle commutes. Since the square on its right-hand side commutes by construction of $s^*\tau$, it suffices to show the commutativity of the square on its left-hand side. Recall that $\psi_{\mathcal{F}}$ was obtained as a relative left adjoint to the morphism of Cartesian fibrations $\phi_{\mathcal{F}} : s^*\mathcal{E} \to t^*\mathcal{E}$ corresponding to the natural transformation $\mathsf{Fun}_{\mathsf{Sm}}(\Delta^1, \mathsf{Span}) \times \Delta^1 \to \mathsf{Span} \to \mathsf{Cat}_\infty$ (postcomposition of the evaluation map with F) and similarly for $\mathcal{G}$. The diagram

$$\begin{array}{ccc} s^*\mathcal{E} & \xrightarrow{\phi_{\mathcal{F}}} & t^*\mathcal{E} \\ \downarrow{\scriptstyle s^*\tau} & & \downarrow{\scriptstyle t^*\tau} \\ s^*\mathcal{H} & \xrightarrow{\phi_{\mathcal{G}}} & t^*\mathcal{H} \end{array}$$

involving the left adjoints $\phi_{\mathcal{F}} \dashv \psi_{\mathcal{F}}$ and $\phi_{\mathcal{G}} \dashv \psi_{\mathcal{G}}$ is commutative by construction, since both composites correspond to the natural transformation

$$\mathsf{Fun}_{\mathsf{Sm}}(\Delta^1, \mathsf{Span}) \underset{t}{\overset{s}{\Downarrow \mathsf{ev}}} \mathsf{Span} \underset{G}{\overset{F}{\Downarrow \tau}} \mathsf{Cat}_\infty.$$

In particular, there is a canonical exchange transformation $\psi_{\mathcal{G}} \circ t^*\tau \Rightarrow s^*\tau \circ \psi_{\mathcal{F}}$ given by the composition

$$\psi_{\mathcal{G}} \circ t^*\tau \Rightarrow \psi_{\mathcal{G}} \circ t^*\tau \circ \phi_{\mathcal{F}} \circ \psi_{\mathcal{F}} \simeq \psi_{\mathcal{G}} \circ \phi_{\mathcal{G}} \circ s^*\tau \circ \psi_{\mathcal{F}} \Rightarrow s^*\tau \circ \psi_{\mathcal{F}}.$$

Consider an edge in $t^*\mathcal{E}$, it takes the form of a diagram similar to (1) in Step 3 of the proof of Theorem 2.18, together with objects $H \in \mathcal{F}(Z')$, $E \in \mathcal{F}(W)$, and a morphism $\kappa : H \to \nabla'_{\otimes} f'^* E$ in $\mathcal{F}(Z')$. The difference compared to the aforementioned Step 3 is that Y' and Z' are not required to be Weil restrictions, and κ need not be an equivalence. On the one hand, the composite $s^*\tau \circ \psi_{\mathcal{F}}$ maps this edge to the edge consisting in the data of the same diagram and the morphism

$$\tau_Z\left(u_\sharp H \xrightarrow{u_\sharp \kappa} u_\sharp \nabla'_{\otimes} f'^* E \xrightarrow{\mathsf{Dis}_{\sharp,\otimes}} \nabla_{\otimes}(\pi_Y)_\sharp \pi_W^* E \xrightarrow{\mathsf{Ex}^*_\sharp} \nabla_{\otimes} f^* g_\sharp E\right)$$

in $\mathcal{G}(Z)$, where the $(-)_\sharp$, $(-)_\otimes$, and $(-)^*$ refer to the functoriality of $\mathcal{F}$. On the other hand, $\psi_{\mathcal{G}} \circ t^*\tau$ maps it to the edge consisting in the data of the same diagram and the morphism

$$u_\sharp \tau_{Z'}(H) \xrightarrow{\tau_{Z'}(\kappa)} u_\sharp \tau_W(\nabla'_{\otimes} f'^*(E)) \simeq u_\sharp \nabla'_{\otimes} f'^* \tau_W(E) \xrightarrow{\mathsf{Dis}_{\sharp,\otimes}} \nabla_{\otimes}(\pi_Y)_\sharp \pi_W^* \tau_W(E) \xrightarrow{\mathsf{Ex}^*_\sharp} \nabla_{\otimes} f^* g_\sharp \tau_W(E),$$

where the $(-)_\sharp$, $(-)_\otimes$, and $(-)^*$ now refer to the functoriality of $\mathcal{G}$. By assumption and by Remark 2.31, τ is compatible with the exchange transformation $\mathsf{Ex}^*_\sharp$ and with $u_\sharp$. For compatibility with the distributivity transformation, note that it is by definition the composition of the exchange transformation $\mathsf{Ex}_{\sharp,\otimes}$ with the counit of the adjunction $(-)_\sharp \dashv (-)^*$. The transformation τ is compatible with them by assumption and Remark 2.31, since compatibility of τ with $(-)_\otimes$ is given by it being a morphism of presheaves of symmetric monoidal categories (naturality for forward morphisms in Span, i.e., spans consisting of one identity and one fold map). This shows that our exchange transformation $\psi_{\mathcal{G}} \circ t^*\tau \Rightarrow s^*\tau \circ \psi_{\mathcal{F}}$ is an equivalence, and, as explained above, this allows us to conclude that Diagram (2) commutes.

Finally, since $t^*\tau$ is a pullback, the induced transformation $(\mathsf{Sm}_\bullet)_{/\mathcal{F}^\simeq} \to (\mathsf{Sm}_\bullet)_{/\mathcal{G}^\simeq}$ is indeed given over $X \in \mathsf{Sm}_{\mathbb{R}}$ by the functor $(\mathsf{Sm}_X)_{/\mathcal{F}^\simeq} \to (\mathsf{Sm}_X)_{/\mathcal{G}^\simeq}$ induced by post-composition by τ (in the sense that a pair $(Y \in \mathsf{Sm}_X, E \in \mathcal{F}(Y))$ is sent to $(Y \in \mathsf{Sm}_X, \tau_Y(E) \in \mathcal{G}(Y))$). This justifies the notation $\tau_\sharp$ in Diagram (2). When $\mathcal{F}$ and $\mathcal{G}$ are subpresheaves of $\mathcal{F}'$ and $\mathcal{G}'$ lifting to sifted-cocomplete categories, the argument is the same as in Step 5 of the proof of Theorem 2.18. □

2.3.2. *Motivic colimit functor associated with the real realization functor.*

We now construct a functor $\mathcal{R}^\omega : \mathsf{Span} \to \mathsf{Cat}_\infty$ satisfying the assumptions of Theorem 2.18, which provides us with a motivic colimit functor related to real realization. Using the naturality result of the previous subsection, we compare it to the motivic Thom spectrum functor. Then, in Subsection 2.3.3, we will compare our new motivic colimit functor to the topological Thom spectrum functor.

Definition 2.32. Let $\mathcal{R}$ be the presheaf of (not necessarily small) symmetric monoidal categories on $\mathsf{Sm}_{\mathbb{R}}$, defined by $X \in \mathsf{Sm}_{\mathbb{R}} \mapsto \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)})$, where functoriality is given by pullback (see Proposition 2.33). Let $\mathcal{R}^\omega : \mathsf{Span} = \mathsf{Span}(\mathsf{Sm}_{\mathbb{R}}, \mathsf{all}, \mathsf{fold}) \to \mathsf{Cat}_\infty$ be the spherical presheaf of essentially small symmetric monoidal categories defined by $X \in \mathsf{Sm}_{\mathbb{R}} \mapsto \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)})^\omega$. Here $\mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)})^\omega$ is viewed as a symmetric monoidal subcategory of $\mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)})$ (as in Proposition 2.11), whose structure is

induced by the Cartesian structure on $\mathsf{Spc}_{/r_{\mathbb{R}}(X)}$ (as a particular case of Lemma B.4, the latter category is equivalent to $\mathcal{P}(*_{/r_{\mathbb{R}}(X)})$ and thus admits finite limits; the symmetric monoidal structure on the stabilization is induced by tensoring with Sp in Pr^{L}, see [Lur17, Ex. 4.8.1.23]).

To apply Definition 2.21 to $\mathcal{F} = \mathcal{R}^{\omega}$, we show the following result.

Proposition 2.33. *The functor $\mathcal{R}$ from Definition 2.32 satisfies all the assumptions of Theorem 2.18 except for essential smallness of the categories in its image, and lifts to sifted-cocomplete categories. It also satisfies a projection formula as in Theorem 2.7(v). The functor $\mathcal{R}^{\omega}$ satisfies all the assumptions of Theorem 2.18; in the statement of the latter, we may choose $\mathcal{F} = \mathcal{R}^{\omega}$ and $\mathcal{G} = \mathcal{R}$. In particular, we obtain a symmetric monoidal motivic colimit functor*

$$M_{\mathcal{R}} : \mathcal{P}_{\Sigma}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega},\simeq} \longrightarrow \mathcal{R}(\mathbb{R}) \simeq \mathsf{Sp}.$$

Proof. **Step 1:** *We prove that $\mathcal{R}$ is a spherical presheaf of symmetric monoidal categories and satisfies assumption (i).* For any morphism $f : X \to Y$ in $\mathsf{Sm}_{\mathbb{R}}$, we have a diagram

$$\begin{array}{ccc} \mathsf{Spc}_{/r_{\mathbb{R}}(X)} & \xrightarrow{\Sigma^{\infty}_{+}} & \mathsf{Sp}\left(\mathsf{Spc}_{/r_{\mathbb{R}}(X)}\right) \\ {\scriptstyle f_\sharp}\downarrow \dashv \uparrow{\scriptstyle f^*} \dashv \downarrow{\scriptstyle f_*} & & {\scriptstyle f_\sharp}\downarrow \dashv \uparrow{\scriptstyle f^*} \\ \mathsf{Spc}_{/r_{\mathbb{R}}(Y)} & \xrightarrow{\Sigma^{\infty}_{+}} & \mathsf{Sp}\left(\mathsf{Spc}_{/r_{\mathbb{R}}(Y)}\right) \end{array}$$

The adjunction $f_\sharp \dashv f^*$ is built as in the case of schemes, where f^* is the pullback functor and $f_\sharp$ is induced by post-composition of structure maps. In particular, f^* preserves products because it is a right adjoint, so it is symmetric monoidal with respect to the Cartesian symmetric monoidal structures on both categories. The right adjoint f_* to f^* exists because pullbacks commute with colimits in spaces. Since both $f_\sharp$ and f^* are left adjoints, they preserve colimits and thus descend to an adjunction on the stabilization of both categories (see [Vol23, Rmk. 2.2], using that stabilization is tensoring with Sp in Pr^{L} by [Lur17, Ex. 4.8.1.23]). The induced functor f^* becomes symmetric monoidal for the smash product. Moreover, we have, for all $X, Y \in \mathsf{Sm}_{\mathbb{R}}$

$$\begin{aligned} \mathcal{R}(X \amalg Y) &\simeq \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X \amalg Y)}) \simeq \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)} \times \mathsf{Spc}_{/r_{\mathbb{R}}(Y)}) \\ &\simeq \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)}) \times \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(Y)}) \simeq \mathcal{R}(X) \times \mathcal{R}(Y). \end{aligned}$$

This proves assumption (i) and establishes $\mathcal{R}$ as a spherical presheaf of symmetric monoidal categories.

Step 2: *We prove that $\mathcal{R}$ satisfies assumption (ii).* Consider a square as in the statement of assumption (ii). We first prove a projection formula, i.e., that for all $S \in \mathcal{R}(X)$ and $T \in \mathcal{R}(Y)$, we have $f_\sharp(S \wedge f^*T) \simeq f_\sharp(S) \wedge T$ (via the canonical map). Since both sides preserve colimits in S and T separately, it suffices to show that the formula holds for the infinite suspension spectra of $S \in \mathsf{Spc}_{/r_{\mathbb{R}}(X)}$ and $T \in \mathsf{Spc}_{/r_{\mathbb{R}}(Y)}$. We have equivalences

$$\begin{aligned} f_\sharp(\Sigma^{\infty}_{+} S \wedge f^*\Sigma^{\infty}_{+} T) &\simeq f_\sharp(\Sigma^{\infty}_{+} S \wedge \Sigma^{\infty}_{+} f^*T) \simeq f_\sharp(\Sigma^{\infty}_{+}(S \times_{r_{\mathbb{R}}(X)} f^*T)) \simeq \Sigma^{\infty}_{+} f_\sharp(S \times_{r_{\mathbb{R}}(X)} f^*T) \\ &\simeq \Sigma^{\infty}_{+}(f_\sharp(S) \times_{r_{\mathbb{R}}(Y)} T) \qquad \text{(pasting law for pullbacks, [Lur09, Lem. 4.4.2.1])} \\ &\simeq f_\sharp(\Sigma^{\infty}_{+} S) \wedge \Sigma^{\infty}_{+} T. \end{aligned}$$

induced by the canonical map (as can be checked at the level of spaces, because it is already the case for the equivalence $f_\sharp(S \times_{r_{\mathbb{R}}(X)} f^*T) \simeq f_\sharp(S) \times_{r_{\mathbb{R}}(Y)} T$).

To prove that the exchange transformation is an equivalence, it again suffices to show it before stabilization. For any arrow $(S \to r_{\mathbb{R}}(X))\mathsf{Spc}_{/r_{\mathbb{R}}(X)}$ with structure map $x : S \to r_{\mathbb{R}}(X)$, we have

$$\begin{aligned} g^* f_\sharp(S \to r_{\mathbb{R}}(X)) &\simeq g^*(S \to r_{\mathbb{R}}(X) \to r_{\mathbb{R}}(Y)) \simeq (S \times_{r_{\mathbb{R}}(Y)} r_{\mathbb{R}}(Y') \to r_{\mathbb{R}}(Y')) \\ &\simeq f'_\sharp(S \times_{r_{\mathbb{R}}(X)} r_{\mathbb{R}}(X') \to r_{\mathbb{R}}(X')) \qquad (\star) \\ &\simeq f'_\sharp g'^*(S \to r_{\mathbb{R}}(X)) \end{aligned}$$

where $(\star)$ follows from the pasting law for pullbacks in the diagram

$$\begin{array}{ccccc} S \times_{r_{\mathbb{R}}(Y)} r_{\mathbb{R}}(Y') & \longrightarrow & r_{\mathbb{R}}(X') & \xrightarrow{f'} & r_{\mathbb{R}}(Y') \\ \downarrow & & {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ S & \xrightarrow{x} & r_{\mathbb{R}}(X) & \xrightarrow{f} & r_{\mathbb{R}}(Y) \end{array}$$

Thus the exchange transformation is an equivalence.

Step 3: *We prove that $\mathcal{R}$ satisfies assumption (iii).* Consider a diagram as in the statement of assumption (*iii*). To lighten notation we deal only with the case $Y = Z^{\amalg n}$; the case of general maps in fold is similar. Write $U := W \times_X Y = \coprod_{i\leq n} U_i$, where U_i lives over the i-th copy Z_i of Z in Y. Let $e : Y' := R_{Y/X}(U) \times_Z Y \to U$ be the natural map; it is given over Z_i by the projection on the i-th component $\prod_{j\leq n} U_j \to U_i$. Then, for any $(A \to r_{\mathbb{R}}(W)) \in \mathsf{Spc}_{/r_{\mathbb{R}}(W)}$, let $B = (\pi_W)^*(A)$, which lives over $r_{\mathbb{R}}(U)$. Write B_i the component of B over $r_{\mathbb{R}}(U_i)$. Now, $e^*(B)$, as a space over $r_{\mathbb{R}}(Y') = \coprod_{i\leq n}(r_{\mathbb{R}}(U_1) \times_{r_{\mathbb{R}}(Z)} \cdots \times_{r_{\mathbb{R}}(Z)} r_{\mathbb{R}}(U_n))$, is given by

$$\coprod_{i\leq n}(r_{\mathbb{R}}(U_1) \times_{r_{\mathbb{R}}(Z)} \cdots \times_{r_{\mathbb{R}}(Z)} B_i \times_{r_{\mathbb{R}}(Z)} \cdots \times_{r_{\mathbb{R}}(Z)} r_{\mathbb{R}}(U_n)).$$

In the remainder of Step 3, unless specified otherwise, all products are fibered over $r_{\mathbb{R}}(Z)$. Then $u_\sharp \nabla'_\otimes f'^*(A) = u_\sharp \nabla'_\otimes e^*(B)$ is given by the fiber product over $r_{\mathbb{R}}(U_1) \times \cdots \times r_{\mathbb{R}}(U_n)$ of the factors $r_{\mathbb{R}}(U_1) \times \cdots \times B_i \times \cdots \times r_{\mathbb{R}}(U_n)$ for $1 \leq i \leq n$. That is, $u_\sharp \nabla'_\otimes e^*(B) = B_1 \times \cdots \times B_n \longrightarrow r_{\mathbb{R}}(Z)$, which coincides with $\nabla_\otimes (\pi_Y)_\sharp (\pi_W)^*(A) = \nabla_\otimes (\pi_Y)_\sharp (B)$. Now $\nabla_\otimes u'_\sharp f'^*(A) = \nabla_\otimes u'_\sharp e^*(B)$ is given by

$$\prod_{i\leq n}(r_{\mathbb{R}}(U_1) \times \cdots \times B_i \times \cdots \times r_{\mathbb{R}}(U_n)) \longrightarrow r_{\mathbb{R}}(Z).$$

The distributivity transformation is now the map

$$B_1 \times \cdots \times B_n \longrightarrow \prod_{i\leq n}(r_{\mathbb{R}}(U_1) \times \cdots \times B_i \times \cdots \times r_{\mathbb{R}}(U_n)) \longrightarrow B_1 \times \cdots \times B_n,$$

which is seen to be the identity by chasing through the construction. Therefore, in our situation, the distributivity transformation is an equivalence.

Step 4: *We show that $\mathcal{R}$ lifts to sifted-cocomplete categories.* For all $X \in \mathsf{Sm}_{\mathbb{R}}$, the slice $\mathsf{Spc}_{/r_{\mathbb{R}}(X)}$ is presentable (as a particular case of Proposition B.4, it is equivalent to $\mathcal{P}(*_{/r_{\mathbb{R}}(X)})$). Then, using [Lur17, Prop. 1.4.4.4], the category $\mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}(X)})$ is cocomplete. Moreover, for any map f in $\mathsf{Sm}_{\mathbb{R}}$, the functor f^* preserves colimits; and for any fold map ∇, the functor $\nabla_\otimes$ preserves sifted colimits, by the same argument as in the proof of Proposition 2.25. Since the operations f^* and $\nabla_\otimes$ describe the functoriality of $\mathcal{R}$ (viewed as a functor on the category Span), the latter lifts to sifted-cocomplete categories.

Step 5: *We prove that $\mathcal{R}^\omega$ satisfies the assumptions of Theorem 2.18.* We proceed as in the proof of Proposition 2.11. In this case, given $X \in \mathsf{Sm}_{\mathbb{R}}$, $\mathcal{R}(X) \cong \mathsf{Sp}(\mathsf{Spc}_{/r_{\mathbb{R}}X})$ is generated under colimits by the (small) set of compact objects $E := \{\Sigma^n \Sigma^\infty_+ V \mid n \in \mathbb{Z}, V \in *_{/r_{\mathbb{R}}X}\}$. Here we identify $\mathsf{Spc}_{/r_{\mathbb{R}}X} \simeq \mathcal{P}(*_{/r_{\mathbb{R}}X})$ via Proposition B.2. The category $*_{/r_{\mathbb{R}}X} \simeq r_{\mathbb{R}}X$ is essentially small as $r_{\mathbb{R}}X$ is a (small) space. The fact that E generates $\mathcal{R}(X)$ under colimits simply follows from the fact that the stabilization is generated under colimits by shifts of infinite suspensions of objects in the original category; and any presheaf is a colimit of representable ones. Object in E are compact because infinite suspensions of objects in the original category are compact by the same argument as in [Hoy14, Prop. C.12]. The remainder of the proof is the same as for Proposition 2.11, using the properties of $\mathcal{R}$ proven in Steps 1 to 4. This uses that the morphisms f^* and $f_\sharp$ commute with infinite suspensions (they are induced at the level of the slice categories before stabilization), and that $f_\sharp$ behaves well with respect to shifts because of the projection formula. □

We now want to apply Proposition 2.30 to a transformation $\alpha : \mathsf{SH}^\omega \to \mathcal{R}^\omega$ of functors $\mathsf{Span} \to \mathsf{Cat}_\infty$. We first define it at the level of the full categories (so not only on the compact objects). To do so, we use the universal property of SH as a presheaf of symmetric monoidal categories. Indeed, as we saw in the construction of the real realization functor, to specify a symmetric monoidal colimit preserving functor $\mathsf{SH}(S) \to \mathcal{C}$ where $\mathcal{C}$ is a symmetric monoidal stable category, it suffices to define a symmetric monoidal functor $\mathsf{Sm}_S^\times \to \mathcal{C}$ which is suitable $\mathbb{A}^1$-invariant and well-behaved with respect to the Nisnevich topology, and inverts $\mathbb{P}^1$. A more precise statement is proven in [Rob13]. This universal property was made into one of the presheaf SH in [DG22]. Before stating it, we need a definition that axiomatizes some functoriality properties of SH we saw in Theorem 2.7.

Definition 2.34 ([DG22])**.** Let $\mathcal{C}$ be an essentially small 1-category with finite limits and a terminal object, and P be a collection of maps in $\mathcal{C}$ stable under pullback and equivalences. A *P-pullback formalism over $\mathcal{C}$* is a functor $C : \mathcal{C}^{\mathsf{op}} \to \mathsf{CAlg}(\widehat{\mathsf{Cat}}_\infty)$ such that:

- for any $f \in P$, $f^* := C(f)$ has a left adjoint $f_\sharp$,
- the exchange transformation is an equivalence for pullback squares with two parallel sides in P (see Theorem 2.7(*vii*)),
- C satisfies the projection formula (see Theorem 2.7(v)).

A *morphism of P-pullback formalisms over $\mathcal{C}$* is a natural transformation $\tau : C \to C'$ such that the canonical transformation $f_\sharp^{C'}\tau \Rightarrow \tau f_\sharp^{C}$ is an equivalence for any $f \in P$. This can be organized into a category $\mathsf{PB}(\mathcal{C}, P)$.

Definition 2.35 ([DG22])**.** Let $\mathsf{PB}(\mathsf{Sm}_\mathbb{R}, \mathsf{Sm})^{c,pt}_{\mathsf{L}_{\mathsf{mot}},(\mathbb{P}^1,\infty)}$ be the subcategory of $\mathsf{PB}(\mathsf{Sm}_\mathbb{R}, \mathsf{Sm})$ consisting of *cocomplete pointed $\mathbb{A}^1$-local Nisnevich-local Sm-pullback formalisms with $\mathbb{P}^1$ invertible*, that is, of objects C such that:

- for any $X \in \mathsf{Sm}_\mathbb{R}$, $C(X)$ is a cocomplete and pointed category,
- C satisfies Nisnevich excision,
- for any $X \in \mathsf{Sm}_\mathbb{R}$, the projection induces a fully faithful functor $C(X) \to C(X \times \mathbb{A}^1)$,
- the essentially unique morphism of pointed cocomplete pullback formalisms $\mathsf{Spc}(-)_* \to C$ sends $(\mathbb{P}^1, \infty) \in \mathsf{Spc}(\mathbb{R})_*$ to an invertible object in $C(\mathbb{R})$.

Theorem 2.36 ([DG22, Cor. 6.35 and Rmk. 7.11])**.** *The functor SH is an initial object in the category $\mathsf{PB}(\mathsf{Sm}_\mathbb{R}, \mathsf{Sm})^{c,pt}_{\mathsf{L}_{\mathsf{mot}},(\mathbb{P}^1,\infty)}$ (see Definition 2.35).*

Lemma 2.37. *The presheaf of symmetric monoidal categories $\mathcal{R}$ is a cocomplete pointed $\mathbb{A}^1$-local Nisnevich-local Sm-pullback formalism over $\mathsf{Sm}_\mathbb{R}$ with $\mathbb{P}^1$ invertible.*

Proof. It follows from Proposition 2.33 that $\mathcal{R}$ is a pullback formalism. Moreover, for any $X \in \mathsf{Sm}_\mathbb{R}$, $\mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(X)}) = \mathcal{R}(X)$ is pointed and cocomplete. The pullback formalism $\mathcal{R}$ is furthermore $\mathbb{A}^1$-local, because $r_\mathbb{R}(X) \simeq r_\mathbb{R}(X \times \mathbb{A}^1)$ for any $X \in \mathsf{Sm}_R$ via the projection, so that $\mathcal{R}(X) \to \mathcal{R}(X \times \mathbb{A}^1)$ is in particular fully faithful. To check that $\mathcal{R}$ has Nisnevich excision, let $\iota : U \hookrightarrow X$ and $p : Y \to X$ form an elementary distinguished Nisnevich square (i.e. ι is an open immersion, and p is an étale map inducing an isomorphism between the complement of U and its preimage by p). Then the corresponding pullback square in $\mathsf{Sm}_\mathbb{R}$ becomes a pushout square in $\mathsf{Spc}(\mathbb{R})$; since $r_\mathbb{R}$ preserves colimits we have

$$\mathsf{Spc}_{/r_\mathbb{R}(X)} \simeq \mathsf{Spc}_{/r_\mathbb{R}(U) \amalg_{r_\mathbb{R}(U \times_X Y)} r_\mathbb{R}(Y)} \simeq \mathsf{Spc}_{/r_\mathbb{R}(U)} \times_{\mathsf{Spc}_{/r_\mathbb{R}(U \times_X Y)}} \mathsf{Spc}_{/r_\mathbb{R}(Y)}$$

by the descent property in Spc. Finally, we have

$$\mathcal{R}(X) \simeq \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(X)}) \simeq \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(U)}) \times_{\mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(U \times_X Y)})} \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(Y)}) \simeq \mathcal{R}(U) \times_{\mathcal{R}(U \times_X Y)} \mathcal{R}(Y)$$

because stabilization preserves limits. Indeed, it is described by tensoring with Sp in Pr^L [Lur17, Ex. 4.8.1.23], but Sp is dualizable [Lur18, Prop. D.7.2.3], thus $- \otimes \mathsf{Sp}$ preserves all limits and colimits; moreover limits in Pr^L are computed on the underlying categories [Lur09, Prop. 5.5.3.13]. Finally, the essentially unique morphism of pointed cocomplete pullback formalisms $\chi : \mathsf{Spc}(-)_* \to \mathcal{R}$ sends $(\mathbb{P}^1, \infty) \in \mathsf{Spc}(\mathbb{R})_*$ to $\Sigma^\infty((r_\mathbb{R}(\mathbb{P}^1) \to r_\mathbb{R}(\mathbb{R}), \infty)) \simeq \Sigma^\infty(\mathbb{S}^1, *) \in \mathsf{Sp}$, which is invertible (see Construction 2.38 below for the description of χ). This verifies all the axioms. □

Construction 2.38. Let $\alpha : \mathsf{SH} \to \mathcal{R}$ be the unique morphism in $\mathsf{PB}(\mathsf{Sm}_\mathbb{R}, \mathsf{Sm})^{c,pt}_{\mathsf{L}_{\mathsf{mot}},(\mathbb{P}^1,\infty)}$ from SH to $\mathcal{R}$ (using Theorem 2.36 and Lemma 2.37). Unraveling the constructions in [DG22, Thm. 6.3 and Rmk. 7.11], we can give an explicit description of the components of α. Given $S \in \mathsf{Sm}_\mathbb{R}$, real realization induces a finite-product-preserving functor $\mathsf{Sm}_S \to \mathsf{Spc}_{/r_\mathbb{R}(S)}$ sending $X \in \mathsf{Sm}_S$ to the real realization of its structure map $r_\mathbb{R}(X) \to r_\mathbb{R}(S)$ (in the terminology of [DG22], this is a morphism of pre-pullback formalisms, for which $\mathsf{Sm}_\bullet$ is initial). The left Kan extension of this functor inverts motivic equivalences, and factors through $\mathsf{Spc}(S)$. The corresponding functor on pointed objects can be post-composed with the infinite suspension functor $(\mathsf{Spc}_{/r_\mathbb{R}(S)})_* \to \mathcal{R}(S)$. There is therefore an induced symmetric monoidal functor $\mathsf{Spc}(S)_* \to \mathcal{R}(S)$ that preserves colimits, and inverts $\mathbb{P}^1$. Finally, this functor factors through $\mathsf{SH}(S)$, and we obtain $\alpha_S : \mathsf{SH}(S) \to \mathcal{R}(S)$.

As promised, we now apply Proposition 2.30 to the transformation α from Construction 2.38.

Proposition 2.39. *The transformation α from Construction 2.38 restricts to a natural transformation $\alpha : \mathsf{SH}^\omega \to \mathcal{R}^\omega$, inducing a commutative diagram of symmetric monoidal categories and symmetric*

monoidal functors

$$\begin{array}{ccc} \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathsf{SH}^{\omega,\simeq}} & \xrightarrow{\alpha_\sharp} & \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^{\omega,\simeq}} \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M_{\mathcal{R}^\omega}} \\ \mathsf{SH}(\mathbb{R}) & \xrightarrow{\alpha_\mathbb{R}} & \mathsf{Sp}\left(\mathsf{Spc}_{/r_\mathbb{R}(\mathbb{R})}\right) \simeq \mathsf{Sp}. \end{array}$$

Proof. To show the first claim, it suffices by Lemma 2.9 to show that for all $X \in \mathsf{Sm}_R$, $U \in \mathsf{Sm}_X$, and $n \in \mathbb{N}$, $\alpha(\Sigma^{-2n,-n}\Sigma^\infty_+ U) \in \mathcal{R}(X)^\omega$. By construction, $\alpha(\Sigma^{-2n,-n}\Sigma^\infty_+ U) \simeq \Sigma^{-n}\Sigma^\infty_+(r_\mathbb{R}U \to r_\mathbb{R}X)$, and the latter is compact (Step 5 of the proof of Proposition 2.33). To prove the remainder of the statement, it suffices to check that $\alpha : \mathsf{SH} \to \mathcal{R}$ satisfies the assumption of Proposition 2.30 (compatibility with the left adjoints $(-)_\sharp$), because then so does its restriction to the subcategories of compact objects. But α is by construction a morphism of pullback formalisms, and so the assumption holds by definition. □

2.3.3. *Comparing the motivic colimit functor $M_{\mathcal{R}^\omega}$ and the topological Thom spectrum functor.*

As mentioned at the beginning of the previous subsection, we now compare $M_{\mathcal{R}^\omega}$ with the topological Thom spectrum functor. We saw in Theorem 1.17 that the unstable real realization functor $r_\mathbb{R} : \mathcal{P}(\mathsf{Sm}_\mathbb{R}) \to \mathsf{Spc}$ was a localization functor, inducing an equivalence between the category of $\mathbb{A}^1$-invariant real-étale sheaves and that of spaces. We now apply this localization to the category $\mathcal{P}(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^{\omega,\simeq}}$.

Lemma 2.40. *The functor $M_{\mathcal{R}^\omega} : \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^{\omega,\simeq}} \to \mathsf{Sp}$ from Proposition 2.33 factors through the localization $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_\mathbb{R}))_{/\mathcal{R}^{\omega,\simeq}}$ of $\mathbb{A}^1$-invariant real-étale sheaves over $\mathcal{R}^{\omega,\simeq}$. In particular, there is a commutative diagram of symmetric monoidal functors*

$$\begin{array}{ccccc} \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathsf{SH}^{\omega,\simeq}} & \xrightarrow{\alpha_\sharp} & \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^{\omega,\simeq}} & \xrightarrow{L_{\mathbb{A}^1,\mathsf{ret}}} & L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_\mathbb{R}))_{/\mathcal{R}^{\omega,\simeq}} \xrightarrow{\simeq} \mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}} \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M_{\mathcal{R}^\omega}} & & \\ \mathsf{SH}(\mathbb{R}) & \xrightarrow{r_\mathbb{R}} & \mathcal{R}(\mathbb{R}) \simeq \mathsf{Sp} & & \end{array} \qquad (3)$$

Proof. By [BEH22, Prop. 2.11], to show that the functor $M_{\mathcal{R}^\omega} : \mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})_{/\mathcal{R}^{\omega,\simeq}} \to \mathsf{Sp}$ factors through $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R}))_{/\mathcal{R}^{\omega,\simeq}}$, it suffices to show that $\mathcal{R}^{\omega,\simeq} \in L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R}))$ is an $\mathbb{A}^1$-invariant real-étale sheaf. We will prove this at the end. Let us first see how this implies the statement of the lemma. Note first that we may ignore the sphericity condition, i.e., the inclusion $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}_\Sigma(\mathsf{Sm}_\mathbb{R})) \subseteq L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_\mathbb{R}))$ is an equivalence, since any real-étale sheaf is in particular spherical (for any $U, V \in \mathsf{Sm}_\mathbb{R}$, the inclusions of U and V in $U \amalg V$ form a real-étale cover of the latter). Moreover, Theorem 1.17 gives an equivalence $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_\mathbb{R}))_{/\mathcal{R}^{\omega,\simeq}} \simeq \mathsf{Spc}_{/r_\mathbb{R}(\mathcal{R}^{\omega,\simeq})}$ induced by real realization, which for $\mathbb{A}^1$-invariant real-étale sheaves corresponds to taking the real sections. Then, $r_\mathbb{R}(\mathcal{R}^{\omega,\simeq}) \simeq \mathsf{Sp}(\mathsf{Spc}_{/r_\mathbb{R}(\mathbb{R})})^{\omega,\simeq} \simeq \mathsf{Sp}^{\omega,\simeq}$, and we obtain Diagram (3), where the symmetric monoidal structure on $\mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}}$ is induced by that of $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_\mathbb{R}))_{/\mathcal{R}^{\omega,\simeq}}$ (described in Lemma 2.42 below).

To conclude the proof, we are left with showing that $\mathcal{R}^{\omega,\simeq}$ is an $\mathbb{A}^1$-invariant real-étale sheaf. We have already proven that $\mathcal{R}$ was an $\mathbb{A}^1$-invariant Nisnevich sheaf (Proposition 2.37). The argument for the real-étale topology is the same, using the fact that the unstable real realization functor inverts the real-étale equivalence by Theorem 1.17. Then, the $\mathbb{A}^1$-invariance of $\mathcal{R}^{\omega,\simeq}$ follows. We defer to Lemma 2.41 just below the proof that $\mathcal{R}^\omega$ is a real-étale sheaf. Then, $\mathcal{R}^{\omega,\simeq}$ is also a real étale sheaf: it is obtained from $\mathcal{R}^\omega$ by postcomposition with the functor $(-)^\simeq : \mathsf{Cat}_\infty \to \mathsf{Spc}$, which preserves limits as it is right adjoint to the inclusion, and being a sheaf is a limit condition. □

Lemma 2.41. *The subpresheaf of compact objects $\mathcal{R}^\omega \subseteq \mathcal{R}$ is a real-étale sheaf on $\mathsf{Sm}_\mathbb{R}$.*

Proof. Recall from the proof of Lemma 2.40 that $\mathcal{R}$ is a real-étale sheaf on $\mathsf{Sm}_\mathbb{R}$. Thus, it suffices to show that for any real-étale cover $\{f_i : U_i \to X\}$ in $\mathsf{Sm}_\mathbb{R}$, if $a \in \mathcal{R}(X)$ satisfies $f_i^*(a) \in \mathcal{R}(U_i)^\omega$, then also $a \in \mathcal{R}(X)^\omega$. Let $\mathsf{colim}_{j\in\mathcal{J}} a_j$ be a cofiltered diagram in $\mathcal{R}(X)$. We want to show that

$$\mathsf{colim}_{j\in\mathcal{J}} \mathsf{Map}_{\mathcal{R}(X)}(a, a_j) \xrightarrow{\simeq} \mathsf{Map}_{\mathcal{R}(X)}(a, \mathsf{colim}_{j\in\mathcal{J}} a_j). \qquad (\star)$$

We will first construct "mapping sheaves" and consider analog of $(\star)$ for these. Since by definition all f_i's are étale, we work over Et_X, the small étale site of X. Let $a, b \in \mathcal{R}(X)$. We can construct a "mapping presheaf" $\underline{\mathsf{Map}}_\mathcal{R}(a, b)$ sending $(f : U \to X) \in \mathsf{Et}_X$ to the space $\mathsf{Map}_{\mathcal{R}(U)}(f^*a, f^*b)$ [Bac22, Cor. 3.8].

Since $\mathcal{R}$ is in particular a real-étale sheaf on Et_X, $\underline{\mathsf{Map}}_{\mathcal{R}}(a,b)$ is also a sheaf. Indeed, if $\mathcal{V} \to Y$ is a real-étale cover in Et_X, then $\mathcal{R}(Y)$ is equivalent to the limit over the Čech complex $\lim \mathcal{R}(\check{C}_\bullet(\mathcal{V}))$, so that

$$\underline{\mathsf{Map}}_{\mathcal{R}}(a,b)(Y) \simeq \mathsf{Map}_{\lim \mathcal{R}(\check{C}_\bullet(\mathcal{V}))}(a,b) \simeq \lim \mathsf{Map}_{\mathcal{R}(\check{C}_\bullet(\mathcal{V}))}(a,b) = \lim \underline{\mathsf{Map}}_{\mathcal{R}}(a,b)(\check{C}_\bullet(\mathcal{V})),$$

where in the second-to-last term, the pullback of a and b to the terms of the Čech complex is implicit. There is a natural map of presheaves $\xi : \mathsf{colim}_{j\in\mathcal{J}}^{\mathcal{P}(\mathsf{Et}_X)} \underline{\mathsf{Map}}_{\mathcal{R}}(a,a_j) \to \underline{\mathsf{Map}}_{\mathcal{R}}(a, \mathsf{colim}_{j\in\mathcal{J}} a_j)$ inducing equivalences after restriction to each U_i. This holds because colimits are computed sectionwise in presheaves, and for any morphism $f : V \to U_i$, the restriction $f^*a \in \mathcal{R}(V)^\omega$ is compact (by assumption $f_i^*a \in \mathcal{R}(U_i)^\omega$ is compact, and f^* preserves compact objects by Proposition 2.33). The right-hand side is already a sheaf, and thus ξ induces an equivalence between the sections on X of the sheafification of the left-hand side, which is equivalent to $\mathsf{colim}_{j\in\mathcal{J}}^{\mathsf{Shv}(\mathsf{Et}_X)} \underline{\mathsf{Map}}_{\mathcal{R}}(a,a_j)$, and that of the right-hand side. Therefore, we have equivalences

$$\begin{aligned}\mathsf{colim}_{j\in\mathcal{J}} \mathsf{Map}_{\mathcal{R}(X)}(a,a_j) =: \mathsf{colim}_{j\in\mathcal{J}} \left(\underline{\mathsf{Map}}_{\mathcal{R}}(a,a_j)(X)\right) &\overset{(\star\star)}{\simeq} \left(\mathsf{colim}_{j\in\mathcal{J}}^{\mathsf{Shv}(\mathsf{Et}_X)} \underline{\mathsf{Map}}_{\mathcal{R}}(a,a_j)\right)(X) \\ &\simeq \underline{\mathsf{Map}}_{\mathcal{R}}(a, \mathsf{colim}_{j\in\mathcal{J}} a_j)(X) := \mathsf{Map}_{\mathcal{R}(X)}(a, \mathsf{colim}_{j\in\mathcal{J}} a_j),\end{aligned}$$

which gives $(\star)$. Here, $(\star\star)$ is obtained by viewing global sections as corepresented by $y(X)$, and using that the (sheafification of the) latter is a compact object in $\mathsf{Shv}(\mathsf{Et}_X)$. To see this, note that it is the terminal object in $\mathsf{Shv}(\mathsf{Et}_X) \simeq \mathsf{Shv}(RX)$ [ES21, Thm. B.10], so it suffices to check that the terminal object in $\mathsf{Shv}(RX)$ is compact. This follows from the fact that RX is a coherent topological space [ABEH25, Prop. 4.1] and the end of the proof of [Lur09, Prop. 6.5.4.4]. □

Lemma 2.42. *The localization $L' : \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega,\simeq}} \to L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))_{/\mathcal{R}^{\omega,\simeq}}$ from Lemma 2.40 is symmetric monoidal, making the right-hand side into a presentably symmetric monoidal subcategory of the left-hand side via the fully faithful right adjoint of the localization L'.*

Notation 2.43. As in Notation 2.20, we denote the corresponding symmetric monoidal structure on $\mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}}$ obtained via the equivalence with $L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))_{/\mathcal{R}^{\omega,\simeq}}$ by $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\omega,\simeq}}$, to distinguish it from the one in Proposition B.3, which we denote by $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\omega,\simeq}}$.

Proof. To lighten notation, let $\mathcal{C} := \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega,\simeq}}$ and $\mathcal{D} := L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))_{/\mathcal{R}^{\omega,\simeq}}$. Recall that there is a source functor (or forgetful functor) $\mathcal{C} \to \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})$ remembering only the domain of an arrow. We first use the criterion of [GGN15, Lem. 3.4]: if we show that local equivalences are preserved by the functors $- \otimes X$ for all $X \in \mathcal{C}$, then this shows that there is an induced symmetric monoidal structure on $\mathcal{D}$, such that L' is symmetric monoidal and its right-adjoint ι is lax symmetric monoidal. This holds because local equivalences in $\mathcal{C}$ are those maps that are equivalences of presheaves after applying the source functor and the $\mathbb{A}^1$-real-étale localization $L : \mathcal{P}(\mathsf{Sm}_{\mathbb{R}}) \to L_{\mathbb{A}^1,\mathsf{ret}}(\mathcal{P}(\mathsf{Sm}_{\mathbb{R}}))$. On the sources, the tensor product of $\mathcal{C}$ is the categorical product (Proposition 2.23). Since L commutes with products (since real realization does), we are done.

As an accessible localization of $\mathcal{C}$ in the sense of [Lur09, §5.5.4], $\mathcal{D}$ is presentable. By symmetric monoidality of L', for any $X, Y \in \mathcal{D}$, we have $X \otimes_{\mathcal{D}} Y = L'(\iota X \otimes_{\mathcal{C}} \iota Y)$. Thus, to show that $\mathcal{D}$ is actually a symmetric monoidal subcategory of $\mathcal{C}$, it suffices to show that such tensor products $\iota X \otimes_{\mathcal{C}} \iota Y$ are already local, and that so is the unit. The latter is clear, and for the former, we use again that being local is checked after applying the source functor; and that $\otimes_{\mathcal{C}}$ is the categorical product on the sources. Since L commutes with products, the product of local objects is local, and this is what we needed.

Finally, we show that the symmetric monoidal structure is presentable. Let $X \in \mathcal{D}$, we have to show that $- \otimes_{\mathcal{D}} X : \mathcal{D} \to \mathcal{D}$ preserves colimits. Let $\mathsf{colim}_i Y_i$ be a colimit diagram in $\mathcal{D}$. Using that both L' and ι preserve the tensor products, and L' preserves colimits, we have:

$$\begin{aligned}(\mathsf{colim}_i Y_i) \otimes_{\mathcal{D}} X &\simeq (\mathsf{colim}_i L'\iota Y_i) \otimes_{\mathcal{D}} X \simeq L'(\mathsf{colim}_i \iota Y_i) \otimes_{\mathcal{D}} L'\iota X \simeq L'((\mathsf{colim}_i \iota Y_i) \otimes_{\mathcal{C}} \iota X) \\ &\simeq L'(\mathsf{colim}_i (\iota Y_i \otimes_{\mathcal{C}} \iota X)) \qquad (\mathcal{C} \text{ is presentably symmetric monoidal}) \\ &\simeq \mathsf{colim}_i L'(\iota(Y_i \otimes_{\mathcal{D}} X)) \simeq \mathsf{colim}_i (Y_i \otimes_{\mathcal{D}} \mathcal{G}).\end{aligned}$$

This finishes the proof. □

The functor M'_{top} can be viewed as the real realization of the motivic Thom spectrum functor. We can now show that it is equivalent to the topological Thom spectrum functor.

Theorem 2.44. *The symmetric monoidal structures* $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\omega,\simeq}}$ *and* $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\omega,\simeq}}$ *agree (see Notation 2.43). The functors* M'_{top} *(from Diagram (3)) and* M_{top} *(from Theorem 2.4) agree as symmetric monoidal functors.*

Proof. By Proposition B.4, the symmetric monoidal structure $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\omega,\simeq}}$ from Proposition B.3 is equivalently the Day convolution structure on $\mathcal{P}(*_{/\mathsf{Sp}^{\omega,\simeq}}) \simeq \mathcal{P}(\mathsf{Sp}^{\omega,\simeq})$, where $\mathsf{Sp}^{\omega,\simeq}$ is viewed as a symmetric monoidal subcategory of Sp with the usual smash product. On the other hand, the underlying category of $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\omega,\simeq}}$ is the same as that of $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\omega,\simeq}}$, namely $\mathcal{P}(\mathsf{Sp}^{\omega,\simeq})$. Then, by Proposition A.2, to show that the symmetric monoidal structure is equivalent to the Day convolution one, it suffices to show that:

(1) the tensor product $\otimes'$ preserves colimits in both variables,
(2) the Yoneda embedding $\mathsf{Sp}^{\omega,\simeq} \to \mathcal{P}(\mathsf{Sp}^{\omega,\simeq})$ admits a symmetric monoidal structure with respect to the smash product on the left-hand side and the structure $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\omega,\simeq}}$ on the right-hand side.

Part (1) was proved in Proposition 2.42. To show (2), consider the commutative diagram of fully faithful embeddings

$$\begin{array}{ccc} \mathcal{P}(*_{/\mathsf{Sp}^{\omega,\simeq}}) \simeq \mathsf{Spc}_{/\mathsf{Sp}^{\omega,\simeq}} & \xhookrightarrow{\iota} & \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega,\simeq}} \simeq \mathcal{P}\big((\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega,\simeq}}\big) \\ {\scriptstyle y_1}\uparrow & & \uparrow{\scriptstyle y_2} \\ \mathsf{Sp}^{\omega,\simeq} \simeq *_{/\mathsf{Sp}^{\omega,\simeq}} & \hookrightarrow & (\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega,\simeq}}. \end{array}$$

The dashed arrow exists because the composition $\iota \circ y_1$ factors through the subcategory of representables (its image only contains arrows with source $y(\mathbb{R}) \in \mathcal{P}(\mathsf{Sm}_{\mathbb{R}})$). We claim that y_2 is symmetric monoidal with respect to the monoidal structures introduced in Theorem 2.18. Indeed, the coCartesian fibration $\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathcal{R}^{\omega,\simeq}} \to \mathsf{Fin}_*$ representing the symmetric monoidal structure is classified by a composition

$$\mathsf{Fin}_* \xrightarrow{H} \mathsf{Cat}_\infty \xrightarrow{\mathcal{P}(-)} \mathsf{Pr}^{\mathsf{L}} \xrightarrow{U} \widehat{\mathsf{Cat}}_\infty$$

where H classifies the coCartesian fibration $(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathcal{R}^{\omega,\simeq}} \longrightarrow \mathsf{Fin}_*$ encoding the symmetric monoidal structure constructed in the proof of Theorem 2.18, and U is the forgetful functor. Let η be the unit of the adjunction $\mathcal{P}(-) \dashv U$. It induces a natural transformation

$$\begin{array}{ccccc} \mathsf{Fin}_* & \xrightarrow{H} & \mathsf{Cat}_\infty & = & \widehat{\mathsf{Cat}}_\infty \\ & & & \overset{U\circ\mathcal{P}(-)}{\underset{\mathcal{P}(-)}{\searrow}} & \overset{\eta}{\Downarrow} \\ & & & & \mathsf{Pr}^{\mathsf{L}} \end{array} \qquad \text{with } U \text{ from } \mathsf{Pr}^{\mathsf{L}} \text{ to } \widehat{\mathsf{Cat}}_\infty$$

between the functors $\mathsf{Fin}_* \longrightarrow \widehat{\mathsf{Cat}}_\infty$ classifying $(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathcal{R}^{\omega,\simeq}}$ and $\mathcal{P}(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathcal{R}^{\omega,\simeq}}$, respectively. This induces a symmetric monoidal functor between these categories, whose underlying functor is by construction the Yoneda embedding y_2, as claimed. By Lemma 2.42, the embedding ι is symmetric monoidal too. Then also y_1 is symmetric monoidal, with respect to the structure $\mathsf{Sp}^{\omega,\simeq,\otimes'}$ (induced by that of $(\mathsf{Sm}_{\mathbb{R}})_{/\mathcal{R}^{\omega,\simeq}}$). Then, the restriction of M'_{top} to $\mathsf{Sp}^{\omega,\simeq}$ is a symmetric monoidal functor $\mathsf{Sp}^{\omega,\simeq,\otimes'} \longrightarrow \mathsf{Sp}^{\otimes}$, where the symmetric monoidal structure on Sp is the usual one. But we also know that the underlying functor is the inclusion. Therefore, $\mathsf{Sp}^{\omega,\simeq,\otimes'}$ is endowed with the symmetric monoidal structure induced from that of Sp. This proves (2).

Thus, the categories $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\omega,\simeq}}$ and $\mathsf{Spc}^{\otimes'}_{/\mathsf{Sp}^{\omega,\simeq}}$ are both equivalent as symmetric monoidal categories to $\mathcal{P}(\mathsf{Sp}^{\omega,\simeq})$ with the Day convolution, where $\mathsf{Sp}^{\omega,\simeq}$ is viewed as a symmetric monoidal subcategory of Sp with the usual smash product. By Proposition A.5, to show that M_{top} and M'_{top} agree as symmetric monoidal functors, it then suffices to show that their restrictions to $\mathsf{Sp}^{\omega,\simeq}$ agree as symmetric monoidal functors. In both cases, this restriction is the embedding of the symmetric monoidal subcategory $\mathsf{Sp}^{\omega,\simeq} \hookrightarrow \mathsf{Sp}$ (see Definition 2.1 and Proposition 2.24). This concludes the proof. □

2.3.4. *The realizations of the algebraic cobordism spectra* MGL*,* MSL*, and* MSp*.*

Theorem 2.45. *The equivalence* $r_{\mathbb{R}} \circ M \simeq M_{\mathsf{top}} \circ r_{\mathbb{R}} \circ \alpha_\sharp$ *from Diagram (3) and Theorem 2.44 induces equivalences of* $\mathcal{E}_\infty$*-rings*

$$r_{\mathbb{R}}\mathsf{MGL} \simeq \mathsf{MO}, \qquad r_{\mathbb{R}}\mathsf{MSL} \simeq \mathsf{MSO}, \qquad r_{\mathbb{R}}\mathsf{MSp} \simeq \mathsf{MU}.$$

Proof. We saw in Example 2.27 that $r_{\mathbb{R}}\mathsf{MGL} = r_{\mathbb{R}}(M(K^\circ \xrightarrow{j} \mathsf{SH}^{\omega,\simeq}))$. By Diagram (3) and Theorem 2.44, this is equivalent as an $\mathcal{E}_\infty$-ring to $M_{\mathsf{top}}(r_{\mathbb{R}}(K^\circ \to \mathsf{SH}^{\omega,\simeq} \to \mathcal{R}^{\omega,\simeq}))$. Recall from Example 2.5 that $\mathsf{MO} = M_{\mathsf{top}}(BO \xrightarrow{j} \mathsf{Sp}^{\omega,\simeq})$. So what we have to show is that $r_{\mathbb{R}}(K^\circ \to \mathsf{SH}^{\omega,\simeq} \to \mathcal{R}^{\omega,\simeq}) \simeq (BO \xrightarrow{j} \mathsf{Sp}^{\omega,\simeq})$ as commutative algebras in $\mathsf{Spc}^{\otimes}_{/\mathsf{Sp}^{\omega,\simeq}}$, or equivalently, as $\mathcal{E}_\infty$-maps with target $\mathsf{Sp}^{\omega,\simeq}$ (by Proposition B.5). Recall that $K^\circ \to \mathsf{SH}^{\omega,\simeq} \to \mathcal{R}^{\omega,\simeq}$ is a morphism of spherical presheaves of $\mathcal{E}_\infty$-spaces, viewed as a commutative algebra in $\mathcal{P}_\Sigma(\mathsf{Sm}_{\mathbb{R}})^{\otimes'}_{/\mathcal{R}^{\simeq}}$ using Proposition 2.28. This is how its real realization is viewed as a map of $\mathcal{E}_\infty$-spaces. In other terms, we have to show that the real realization of the motivic j-homomorphism, followed by $\alpha : \mathsf{SH}^{\simeq} \to \mathcal{R}^{\simeq}$, is the topological j-homomorphism as a map of $\mathcal{E}_\infty$-spaces.

This follows from the constructions of the topological and motivic j-homomorphisms as symmetric monoidal functors in Examples 2.5 and 2.27. Indeed, the various steps of the construction, starting from (∞-)groupoids of vector bundles and then extending to the group completion, correspond to each other under real realization. To see this, recall that $r_{\mathbb{R}}(BGL_n) \simeq BO_n$, and that $K^\circ \simeq \mathsf{L}_{\mathsf{mot}}BGL$ by Example 2.27, so that $r_{\mathbb{R}}(K^\circ) \simeq BO$ by similar computations as in Lemma 3.7.

The proof follows in the exact same way for MSL and MSp, using Example 2.27 and the fact that $r_{\mathbb{R}}(KSL^\circ) = BSO$ and $r_{\mathbb{R}}(KSp^\circ) = BU$ (with the same proof as in the case of K°). □

3. The real realizations of some classical motivic spectra

In this section, we compute the real realizations of several classical motivic spectra, either at the level of spectra, $\mathcal{E}_1$- or $\mathcal{E}_\infty$-rings : algebraic K-theory KGL and its very effective cover kgl, motivic cohomology $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$, their variant $\widetilde{\mathsf{H}\mathbb{Z}}$, and Hermitian K-theory KO. All of them will appear in our computations with ko in Section 3.4. The complex realizations of these motivic spectra behave "as expected": for example, $r_{\mathbb{C}}(\mathsf{KGL}) \simeq \mathsf{KU}$ and $r_{\mathbb{C}}(\mathsf{kgl}) \simeq \mathsf{KU}_{\geq 0}$ [ARØ20, Lem. 2.3], $r_{\mathbb{C}}(\mathsf{H}\mathbb{Z}) \simeq \mathsf{H}\mathbb{Z}$ [Lev14, in particular Thm. 5.5], and $r_{\mathbb{C}}(\mathsf{KO}) \simeq \mathsf{KO}^{\mathsf{top}}$ [ARØ20, Lem. 2.13]. As we will see, real realization holds more surprises for us. In the whole section, let $k \hookrightarrow \mathbb{R}$ be a fixed subfield (in order to define a real realization functor on $\mathsf{SH}(k)$).

3.1. The real realizations of $\mathsf{H}\mathbb{Z}/2$ and $\mathsf{H}\mathbb{Z}$.

The motivic spectra $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$ appear as very effective slices of ko, whose real realizations we will need in Section 4.

Definition 3.1. The *motivic Eilenberg-Mac Lane spectrum* $\mathsf{H}\mathbb{Z} \in \mathsf{SH}(k)$, is equivalently defined as:

(i) The motivic spectrum representing the motivic cohomology theory in $\mathsf{SH}(k)$, i.e., for all $X \in \mathsf{Sm}_k$ and $p, q \in \mathbb{Z}$, we have $H^{p,q}(X, \mathbb{Z}) \cong [\Sigma^\infty_+ X, \Sigma^{p,q}\mathsf{H}\mathbb{Z}]$.

(ii) The zeroth effective slice of the sphere spectrum in $\mathsf{SH}(k)$.

(iii) The effective cover of *Milnor K-theory*: $\mathsf{H}\mathbb{Z} = f_0\underline{K}^M_*$ [Mor99, §6.3].

Definitions (i) and (ii) are equivalent by [Lev08], and definition (iii) is equivalent to (i) and (ii) by [Bac17, Lem. 12]. The *motivic Eilenberg-Mac Lane spectrum* $\mathsf{H}\mathbb{Z}/2$ is defined as the cofiber of the multiplication by 2 map on $\mathsf{H}\mathbb{Z}$ (it represents mod 2 motivic cohomology and is equivalent to $f_0(\underline{K}^M_*/2)$).

Remark 3.2. Definition (iii) allows us to define an $\mathcal{E}_\infty$-structure on $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$. Indeed, since the functor $f_0 : \mathsf{SH}(k)^\heartsuit \to \mathsf{SH}(k)^{\mathsf{eff}}$ is lax symmetric monoidal by Proposition 1.8, we only have to show $\underline{K}^M_*, \underline{K}^M_*/2 \in \mathsf{CAlg}(\mathsf{SH}(k)^\heartsuit)$. By [Lur17, Rmk. 1.2.1.12], the heart of a t-structure is a 1-category, and therefore its commutative algebras are the commutative algebras in the 1-categorical sense. By [Mor99, after Thm. 6.4.7], we have $\underline{K}^{MW}_* \simeq \underline{\pi}_0(\mathcal{S})_* \in \mathsf{SH}(k)^\heartsuit$. Since the truncation functor $\mathsf{SH}(k)_{\geq 0} \to \mathsf{SH}(k)^\heartsuit$ is symmetric monoidal [AN21, Lem. A.12], the image $\underline{\pi}_0(\mathcal{S})_*$ of $\mathcal{S} \in \mathsf{CAlg}(\mathsf{SH}(k)_{\geq 0})$ inherits the structure of a commutative algebra object. Because $\underline{K}^M_*$ and $\underline{K}^M_*/2$ are quotients of $\underline{K}^{MW}_*$, they also define commutative algebras.

Proposition 3.3. *The motivic spectra* $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$, *and* $\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}^{MW}_*$ *are very effective.*

Proof. Since the functor $r_0 : \mathsf{SH}(k) \to \mathsf{SH}(k)^{\mathsf{eff}}$ is t-exact (Proposition 1.8), and $\mathsf{SH}(k)^{\mathsf{eff}}_{\geq 0} = \mathsf{SH}(k)^{\mathsf{veff}}$ by Proposition 1.5(i), it suffices to show that $\underline{K}^{MW}_*$, $\underline{K}^M_*$, and $\underline{K}^M_*/2$ belong to the connective part in the homotopy t-structure on $\mathsf{SH}(k)$. Actually, they even belong to the heart: by [Bac17, Lem. 6 and below], they are effective homotopy modules (i.e., they belong to the image of $\iota_0^\heartsuit : \mathsf{SH}(k)^{\mathsf{eff},\heartsuit} \to \mathsf{SH}(k)^\heartsuit$). □

When trying to compute real realizations, the case of $\mathsf{H}\mathbb{Z}/2$ is easier to deal with than that of $\mathsf{H}\mathbb{Z}$, because the global sections of the homotopy sheaves of $\mathsf{H}\mathbb{Z}/2$ are known.

Definition 3.4. Let $\tau \in H^{0,1}(k, \mathbb{Z}/2) \cong \mu_2(k) \cong \mathbb{Z}/2$ be a generator (μ_2 denotes the square roots of unity), i.e., corresponding to $-1 \in \mu_2(k)$.

Theorem 3.5 ([Voe96])**.** *There are isomorphism of graded rings*

$$\underline{K}^M_*(k)/2 \cong H^*_{\acute{e}t}(k, \mathbb{Z}/2) \cong \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_*(k).$$

Moreover, $\underline{\pi}_{*,*}(\mathsf{H}\mathbb{Z}/2)(\mathbb{R}) \simeq \mathbb{Z}/2[\tau, \rho]$*, where* τ *has bidegree* $(0, -1)$ *and* ρ *has bidegree* $(-1, -1)$*.*

Proof. Milnor's conjecture, proven by Voevodsky [Voe96], states in particular that for a field k' of characteristic not 2, we have $\underline{K}^M_*(k')/2 \cong H^*_{\acute{e}t}(k', \mathbb{Z}/2)$. In words, the Milnor K-theory $\mathsf{mod}\,2$ of k' is equal to the étale cohomology $\mathsf{mod}\,2$ of k', or equivalently the group cohomology of the absolute Galois group of k' with coefficients in $\mathbb{Z}/2$. By Voevodsky's proof we also have $\underline{K}^M_n(k')/2 \cong H^{n,n}(k', \mathbb{Z}/2)$ (where the right hand-side is motivic cohomology mod 2, so it is $\underline{\pi}_{-n,-n}(\mathsf{H}\mathbb{Z}/2)(k)$ if $k = k'$), and that cup product with $\tau \in H^{0,1}(k, \mathbb{Z}/2)$, induces isomorphisms $H^{p,q}(k', \mathbb{Z}/2) \cong H^{p,q+1}(k', \mathbb{Z}/2)$ when $0 \leq p \leq q$ (see for example [RØ16, Lem. 6.1 and the end of the proof of Lemma 6.9], or [KRØ20, (7.1)]).

In particular, if $k = k'$, we get $\underline{K}^M_n(k)/2 \cong [\mathsf{Spec}(k), \Sigma^{n,n}\mathsf{H}\mathbb{Z}/2] \cong \underline{\pi}_{-n,-n}(\mathsf{H}\mathbb{Z}/2)(k)$. If $k = \mathbb{R}$, by the above $\underline{K}^M_*(\mathbb{R})/2$ is the group cohomology $\mathsf{mod}\,2$ of $\mathbb{Z}/2$ (for the trivial action). By [Bro94, §III.1, Ex. 2], it is thus equal to $\mathbb{Z}/2$ in every non-negative degree (and trivial otherwise), and as a ring it is polynomial on one generator, namely ρ. Finally, the last claim follows from the statement about multiplication by $\tau \in \underline{\pi}_{0,-1}(\mathsf{H}\mathbb{Z}/2)(\mathbb{R})$. □

Proposition 3.6. *There is an equivalence of* $\mathcal{E}_1$*-maps*

$$r_{\mathbb{R}}(\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2) \simeq (\mathsf{H}\mathbb{Z}/2[t^2] \to \mathsf{H}\mathbb{Z}/2[t])$$

where $\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$ *is the projection, and* $\mathsf{H}\mathbb{Z}/2[t^2] \to \mathsf{H}\mathbb{Z}/2[t]$ *is the map of free* $\mathcal{E}_1$*-*$\mathsf{H}\mathbb{Z}/2$*-algebras (see Definition 1.20) inducing the inclusion* $\mathbb{Z}/2[t^2] \subseteq \mathbb{Z}/2[t]$ *in homotopy.*

Proof. We proceed in three steps. We first identify the homotopy groups of the real realization of $\mathsf{H}\mathbb{Z}/2$ and their structure as a graded ring. Then, using this and a theorem of Hopkins and Mahowald, we identify $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ as a free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebra. Finally, we deduce the homotopy ring and the $\mathcal{E}_1$-structure of $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z})$ from the cofiber sequence $\mathsf{H}\mathbb{Z} \xrightarrow{\cdot 2} \mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$.

Step 1: *Computation of the homotopy ring of* $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$*.* By Lemma 1.14, $\pi_k(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$ is obtained as the colimit $\mathsf{colim}_n\, \underline{\pi}_k(\mathsf{H}\mathbb{Z}/2)_n(\mathbb{R}) = \mathsf{colim}_n\, \pi_{k-n,-n}(\mathsf{H}\mathbb{Z}/2)(\mathbb{R})$. By Theorem 3.5, for $k \geq 0$ and $n \geq k$ this can be rewritten as

$$\mathsf{colim}_n H^{n-k,n}(\mathbb{R}, \mathbb{Z}/2) \cong \mathsf{colim}_n \tau^k H^{n-k,n-k}(\mathbb{R}, \mathbb{Z}/2) \cong \mathsf{colim}_n \tau^k \underline{K}^M_{n-k}(\mathbb{R}) \cong \mathsf{colim}_n \mathbb{Z}/2\{\tau^k \rho^{n-k}\}.$$

This colimit is $\mathbb{Z}/2$ with generator $(\tau\rho^{-1})^k \in \underline{\pi}_k(\mathsf{H}\mathbb{Z}/2)_0(\mathbb{R})$. It follows that $\pi_*(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2[t]$ is a polynomial ring on a single generator $t := r_{\mathbb{R}}(\tau\rho^{-1})$ in degree 1.

Step 2: *Structure of* $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ *as an* $\mathcal{E}_1$*-ring.* We will show that there exists a map of $\mathcal{E}_2$-rings $\mathsf{H}\mathbb{Z}/2 \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$. Here, on the left-hand side, $\mathsf{H}\mathbb{Z}/2$ is the topological Eilenberg-Mac Lane spectrum, whereas on the right-hand side it is the motivic Eilenberg-Mac Lane spectrum from Definition 3.1. By Proposition 1.21(*iii*), we obtain a map $\mathsf{H}\mathbb{Z}/2[t] \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ sending t on the left hand-side to the element $t \in \pi_1(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2))$ we had above. Then, since by Proposition 1.21(*ii*), $\pi_*(\mathsf{H}\mathbb{Z}/2[t]) \cong \mathbb{Z}/2[t]$ as rings, and $\pi_*(\mathsf{H}\mathbb{Z}/2[t]) \to \pi_*(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}))$ is a ring homomorphism, we deduce that the latter is an isomorphism, so $\mathsf{H}\mathbb{Z}/2[t] \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$ is an equivalence of $\mathcal{E}_1$-rings.

To produce the aforementioned $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}/2 \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$, we appeal to the so-called Hopkins–Mahowald theorem ([Mah79], reformulated in [MNN15, Thm. 4.18]): the free $\mathcal{E}_2$-ring with $2 = 0$ is $\mathsf{H}\mathbb{Z}/2$, i.e., there is a pushout diagram of $\mathcal{E}_2$-rings

$$\begin{array}{ccc} F_{\mathcal{E}_2}(\mathbb{S}^0) & \xrightarrow{2} & \mathbb{S} \\ {\scriptstyle 0}\downarrow & \ulcorner & \downarrow \\ \mathbb{S} & \longrightarrow & \mathsf{H}\mathbb{Z}/2. \end{array}$$

Since $2 = 0 \in \pi_*(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$ by the previous computation, we obtain an $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}/2 \to r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}/2)$.

Step 3: *Realization of* $\mathsf{H}\mathbb{Z}$*, and of the quotient map to* $\mathsf{H}\mathbb{Z}/2$*.* Consider once more the cofiber sequence $\mathsf{H}\mathbb{Z} \xrightarrow{\cdot 2} \mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$. We claim that $r_{\mathbb{R}}(\cdot 2) = \cdot 2 = 0$ on $r_{\mathbb{R}}\mathsf{H}\mathbb{Z}$. By definition, we have $\mathsf{H}\mathbb{Z} = f_0\underline{K}^M_*$. However, in Milnor K-theory, $2\rho = 0$. Indeed, $\rho \in [\mathcal{S}^0, \mathbb{G}_m] \cong \underline{\pi}_0(\mathcal{S})_1(k) \cong K^{MW}_1(k)$ corresponds to $[-1]$

in the Milnor-Witt K-theory of k [Mor99, Thm. 6.3.3 and 6.4.1]. But then in Milnor K-theory, we obtain $2\rho = [-1] + [-1] = [(-1)^2] = [1] = 0$. Thus, after inverting ρ, the multiplication by 2 map becomes the zero map. We obtain a long exact sequence (here $n \geq 0$)

$$\cdots \longrightarrow \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \xrightarrow{\cdot 2=0} \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \xrightarrow{q_{n+1}} \pi_{n+1}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2$$
$$\pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \xrightarrow[\cdot 2=0]{} \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \xrightarrow[q_n]{} \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2 \longrightarrow \cdots .$$

By Lemma 1.11, all spectra involved are connective (since $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$ are very effective by Proposition 3.3). Therefore, q_0 is also surjective; it is an isomorphism. Then $\pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$ and ∂_0 is a surjection $\mathbb{Z}/2 \to \mathbb{Z}/2$, namely an isomorphism. Thus, the injective map q_1 has trivial image, and so $\pi_1(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) = 0$. We are back to the original situation: the long exact sequence repeats in exactly the same way. Therefore, $\pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z})$ is $\mathbb{Z}/2$ if n is even and non-negative and 0 else. Let u be a generator of $\pi_2(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$. Since q_2 is an isomorphism, the real realization of the quotient map sends u to the generator t^2 of $\pi_2(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$. Since this quotient map induces after real realization a ring homomorphism on homotopy, and an isomorphism in even degrees, we obtain $q_{2n}(u^n) = t^{2n}$, so u^n generates $\pi_{2n}(r_{\mathbb{R}}\mathsf{H}\mathbb{Z})$. This proves that $\pi_*(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2[t^2]$ as rings. The identification with $\mathsf{H}\mathbb{Z}/2[t^2]$ as $\mathcal{E}_1$-rings follows as in Step 2 from the Hopkins–Mahowald theorem. □

3.2. The real realizations of KGL and kgl.

The algebraic K-theory spectrum KGL (as defined in [BH20, §3.2] for example) appears in our computations with KO because the two of them are related by the Wood cofiber sequence $\Sigma^{1,1}\mathsf{KO} \xrightarrow{\eta} \mathsf{KO} \xrightarrow{c} \mathsf{KGL}$ where $\eta : \mathbb{G}_m \to \mathbb{1}$ is the motivic Hopf map and c is the forgetful map (Hermitian K-theory is defined using bundles with a symmetric form, and algebraic K-theory all algebraic bundles, the forgetful map ignores the form) [RØ16, Thm. 3.4].

Lemma 3.7 ([BH20, Lem. 3.9])**.** *We have $r_{\mathbb{R}}\mathsf{KGL} \simeq 0$.*

Proof. Let us show that $\pi_n(r_{\mathbb{R}}\mathsf{KGL}) = 0$ for all $n \in \mathbb{Z}$. We want to use Lemma 1.12 about computing real realization of spectra levelwise. Since $\mathsf{KGL} = (K, K, \dots)$ where K is the motivic localization of $\mathbb{Z} \times BGL$ [MV99], we first compute

$$r_{\mathbb{R}}(K) = r_{\mathbb{R}}(\mathsf{L}_{\mathsf{mot}}(\mathbb{Z} \times BGL)) \simeq r_{\mathbb{R}}\left(\bigsqcup_{\mathbb{Z}} BGL\right) \simeq \bigsqcup_{\mathbb{Z}} r_{\mathbb{R}}(BGL) \underset{(\star)}{\simeq} \mathbb{Z} \times B\left(r_{\mathbb{R}}\left(\bigcup_{n\geq 1} GL_n\right)\right)$$
$$\simeq \mathbb{Z} \times B\left(\bigcup_{n\geq 1} GL_n(\mathbb{R})\right) \underset{(\star\star)}{\simeq} \mathbb{Z} \times B\left(\bigcup_{n\geq 1} O_n\right) \simeq \mathbb{Z} \times BO.$$

Here $(\star\star)$ follows from the fact that $GL_n(\mathbb{R})$ deformation retracts onto the orthogonal group $O_n(\mathbb{R})$ by Gram-Schmidt orthogonalization, and $(\star)$ from the fact that $r_{\mathbb{R}}$ commutes with colimits and finite products, in particular with geometric realizations and the bar construction. Then, by Lemma 1.12, $\pi_n(r_{\mathbb{R}}\mathsf{KGL}) \cong \mathsf{colim}_m\, \pi_{n+m}(\mathbb{Z} \times BO)$ for all $n \in \mathbb{Z}$. The homotopy groups of $\mathbb{Z} \times BO$ are given by $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \dots$ repeating with period 8 [Bot59]. In particular, there is a cofinal subsequence of zeroes appearing in the colimit, so all homotopy groups of $r_{\mathbb{R}}\mathsf{KGL}$ vanish. □

Proposition 3.8. *There is an equivalence of $\mathcal{E}_1$-maps*

$$r_{\mathbb{R}}(\mathsf{kgl} \to \mathsf{H}\mathbb{Z}) \simeq (\mathsf{H}\mathbb{Z}/2[t^4] \to \mathsf{H}\mathbb{Z}/2[t^2])$$

where $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ is the cofiber of the map $\beta_{\mathsf{kgl}} : \mathbb{P}^1 \wedge \mathsf{kgl} \to \mathsf{kgl}$ induced by the periodicity generator β_{KGL} for KGL, and $\mathsf{H}\mathbb{Z}/2[t^4] \to \mathsf{H}\mathbb{Z}/2[t^2]$ is the map of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras (see Definition 1.20) inducing the inclusion $\mathbb{Z}/2[t^4] \subseteq \mathbb{Z}/2[t^2]$ in homotopy.

Proof. We consider the cofiber sequence $T \wedge \mathsf{kgl} \to \mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ of the statement [ARØ20, Prop. 2.7]. By Proposition 3.6, $r_{\mathbb{R}}(\mathsf{H}\mathbb{Z}) \simeq \mathsf{H}\mathbb{Z}/2[t^2]$. We defer to Lemma 3.9 below the proof that $r_{\mathbb{R}}(\beta_{\mathsf{kgl}}) = 0$. Using this, the long exact sequence induced in homotopy after real realization splits as short exact sequences

$$0 \longrightarrow \pi_n(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \longrightarrow \pi_{n-2}(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow 0$$

for all $n \in \mathbb{Z}$. For n odd, the middle term vanishes, so all odd homotopy groups of $r_{\mathbb{R}}\mathsf{kgl}$ are trivial. By Lemma 1.11, $r_{\mathbb{R}}\mathsf{kgl}$ is connective, so for $n = 0$, we obtain an isomorphism $\pi_0(r_{\mathbb{R}}\mathsf{kgl}) \to \pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$.

For $n = 2$, the map on the right of the sequence is therefore a surjection $\mathbb{Z}/2 \to \mathbb{Z}/2$, so it is an isomorphism, and $\pi_2(r_{\mathbb{R}}\mathsf{kgl}) = 0$. For $n = 4$ we obtain an isomorphism $\pi_n(r_{\mathbb{R}}\mathsf{kgl}) \to \pi_n(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}) \cong \mathbb{Z}/2$. Continuing this argument inductively, $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ induces after real realization isomorphisms in homotopy in degrees divisible by 4, and the other homotopy groups of $r_{\mathbb{R}}\mathsf{kgl}$ vanish. The fact that $r_{\mathbb{R}}\mathsf{kgl} \simeq \mathsf{H}\mathbb{Z}/2[t^4]$ as $\mathcal{E}_1$-rings is then proven exactly as in Proposition 3.6 for $r_{\mathbb{R}}\mathsf{H}\mathbb{Z}$. In this construction, using the Hopkins–Mahowald theorem, the realization of the $\mathcal{E}_\infty$-map $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ becomes a map of $\mathcal{E}_2$-rings under $\mathsf{H}\mathbb{Z}/2$, in particular a map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras (by the proof of Proposition 1.19(*iii*)). Then, $\mathsf{H}\mathbb{Z}/2[t^4]$ being a free object, the real realization of $\mathsf{kgl} \to \mathsf{H}\mathbb{Z}$ is exactly the map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebras in the statement. □

Lemma 3.9. *In the Notation of Proposition 3.8, we have* $r_{\mathbb{R}}(\beta_{\mathsf{kgl}}) = 0$.

Proof. Let us abbreviate $\beta := \beta_{\mathsf{kgl}}$. After real realization, the map $T \wedge \mathsf{kgl} \to \mathsf{kgl}$ induces multiplication by $r_{\mathbb{R}}(\beta)$ in homotopy, viewing $r_{\mathbb{R}}(\beta)$ as an element in $\pi_1(r_{\mathbb{R}}\mathsf{kgl})$ (because $\beta : T \to \mathsf{kgl}$ and $r_{\mathbb{R}}(T) = \mathbb{S}^1$). We want to show that $r_{\mathbb{R}}(\beta)\cdot 1 = 0$ where $1 \in \pi_0(r_{\mathbb{R}}\mathsf{kgl})$, i.e., that the map $\pi_0(r_{\mathbb{R}}\mathsf{kgl}) \to \pi_1(r_{\mathbb{R}}\mathsf{kgl})$ induced by β is the zero map. Consider the cofiber sequence appearing as the last row in the following diagram (obtained by the octahedral axiom)

$$\begin{array}{ccccc}
\Sigma^{4,2}\mathsf{kgl} & = & \Sigma^{4,2}\mathsf{kgl} & \xrightarrow{\Sigma^{2,1}\beta} & \Sigma^{2,1}\mathsf{kgl} \\
{\scriptstyle \Sigma^{2,1}\beta}\downarrow & & {\scriptstyle \beta^2}\downarrow & & {\scriptstyle \beta}\downarrow \\
\Sigma^{2,1}\mathsf{kgl} & \xrightarrow{\beta} & \mathsf{kgl} & = & \mathsf{kgl} \\
{\scriptstyle \Sigma^{2,1}\gamma}\downarrow & & \downarrow & & {\scriptstyle \gamma}\downarrow \\
\Sigma^{2,1}\mathsf{H}\mathbb{Z} & \dashrightarrow & \mathsf{kgl}/\beta^2 & \dashrightarrow & \mathsf{H}\mathbb{Z}.
\end{array}$$

Then we have a commutative diagram

$$\begin{array}{ccccccc}
 & & & & \pi_0(r_{\mathbb{R}}\mathsf{kgl}) & \xrightarrow{\beta} & \pi_1(r_{\mathbb{R}}\mathsf{kgl}) \\
 & & & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \pi_2(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) & \xrightarrow{\partial} & \pi_0(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) & \longrightarrow & \pi_1(r_{\mathbb{R}}(\mathsf{kgl}/\beta^2)) \longrightarrow \cdots
\end{array} \tag{4}$$

where the bottom row is the long exact sequence induced by this cofiber sequence after realization, and the square comes from the left bottom square in the previous diagram.

Step 1: *It suffices to show that ∂ is surjective; in other terms, that it hits* $1 \in \pi_0(\mathsf{kgl}/\beta) \cong \mathbb{Z}/2$. Indeed, note that the proof of Proposition 3.8 did not use that $r_{\mathbb{R}}(\beta) = 0$ to obtain the isomorphisms $\pi_0(r_{\mathbb{R}}\mathsf{kgl}) \cong \pi_0(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) \cong \pi_0(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z})) \cong \mathbb{Z}/2$. This fact implies that the vertical map on the left-hand side in Diagram (4) is an isomorphism. The vertical map on the right-hand side is an isomorphism, because in the long exact sequence induced by the real realization of the cofiber sequence $\Sigma^{4,2}\mathsf{kgl} \xrightarrow{\beta^2} \mathsf{kgl} \to \mathsf{kgl}/\beta^2$, the groups $\pi_i(r_{\mathbb{R}}(\Sigma^{4,2}\mathsf{kgl})) \cong \pi_{i-2}(r_{\mathbb{R}}\mathsf{kgl})$ vanish for $i = 0, 1$ by connectivity of $r_{\mathbb{R}}\mathsf{kgl}$. Thus, in Diagram (4), by exactness, if ∂ is surjective then β is the zero map.

Step 2: *Reduction to a computation in the homotopy of* $\mathsf{H}\mathbb{Z}/2 \in \mathsf{SH}(k)$. By Step 1, it suffices to show that $\partial(t^2) = 1$, where t^2 is viewed as an element in $\pi_*(r_{\mathbb{R}}(\mathsf{kgl}/\beta)) \cong \pi_*(r_{\mathbb{R}}(\mathsf{H}\mathbb{Z})) \cong \mathbb{Z}/2[t^2]$. By Proposition 3.6, the quotient map $\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2$ induces isomorphisms in even homotopy groups after real realization. Let $d : \mathsf{H}\mathbb{Z} \to \Sigma^{3,1}\mathsf{H}\mathbb{Z}$ be the boundary map representing ∂ before real realization. We claim: (1) that the diagram

$$\begin{array}{ccc}
\mathsf{H}\mathbb{Z} & \xrightarrow{d} & \Sigma^{3,1}\mathsf{H}\mathbb{Z} \\
{\scriptstyle \mathrm{mod}\,2}\downarrow & & \downarrow{\scriptstyle \mathrm{mod}\,2} \\
\mathsf{H}\mathbb{Z}/2 & \xrightarrow{\mathsf{Sq}^3} & \Sigma^{3,1}\mathsf{H}\mathbb{Z}/2
\end{array}$$

is commutative, where Sq^3 is the third motivic Steenrod square (see [Voe03]); and (2) that $\mathsf{Sq}^3(\tau^2) = \rho^3$. Claims (1) and (2) imply that $\partial(t^2) = 1$. Indeed, $t^2 \in \pi_2(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2) \cong \mathsf{colim}_n\, \underline{\pi}_2(\mathsf{H}\mathbb{Z}/2)_n(\mathbb{R})$ corresponds to $\rho^{-2}\tau^2 \in \underline{\pi}_2(\mathsf{H}\mathbb{Z}/2)_0(\mathbb{R})$ (see Theorem 3.5), or equivalently, in the colimit, to $\tau^2 \in \underline{\pi}_2(\mathsf{H}\mathbb{Z}/2)_2(\mathbb{R})$. It is therefore mapped to $\rho^3 \in \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_3(\mathbb{R})$, corresponding in $\mathsf{colim}_n\, \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_n(\mathbb{R}) \cong \pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$ to $1 \in \underline{\pi}_0(\mathsf{H}\mathbb{Z}/2)_0(\mathbb{R})$ and thus a generator of $\mathbb{Z}/2 \cong \pi_0(r_{\mathbb{R}}\mathsf{H}\mathbb{Z}/2)$. This suffices because in the commutative square of Claim (1), the vertical maps induce isomorphisms on the even homotopy groups after real

realization.

Step 3: *We prove Claim* (1). By [RØ16, Lem. A.4], we have

$$[\mathsf{H}\mathbb{Z}, \Sigma^{3,1}\mathsf{H}\mathbb{Z}/2] = H^{0,0}(\mathsf{Spec}(\mathbb{R}), \mathbb{Z}/2)\{\mathsf{Sq}^3 \circ (\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2)\} = \mathbb{Z}/2\{\mathsf{Sq}^3 \circ (\mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}/2)\}$$

because $H^{0,0}(\mathsf{Spec}(\mathbb{R}), \mathbb{Z}/2) \cong \underline{K}_0^M/2(\mathbb{R}) \cong \mathbb{Z}/2$ by Theorem 3.5. Then, $(\mathsf{mod}\,2)\circ d$ is either $\mathsf{Sq}^3 \circ (\mathsf{mod}\,2)$ or 0. To distinguish between them, we take complex realizations. The complex realization of the sequence $\Sigma^{2,1}\mathsf{kgl} \to \mathsf{kgl} \xrightarrow{\gamma} \mathsf{H}\mathbb{Z}$ is the sequence $\Sigma^2\mathsf{ku} \to \mathsf{ku} \to \mathsf{H}\mathbb{Z}$ (by [Lev14, in particular Thm. 5.5] for $\mathsf{H}\mathbb{Z}$ and [ARØ20, Lem. 2.3] for kgl). Therefore, the complex realization of d is exactly the suspension of the map $\Sigma^{-1}\mathsf{H}\mathbb{Z} \to \Sigma^2\mathsf{H}\mathbb{Z}$ called the first k-invariant of ku. This map is the non-zero operation Q_1 (whose reduction $\mathsf{mod}\,2$ is non-zero as well), see [Bru21, Section 2] for a proof and more details.

Step 4: *We prove Claim* (2). By [Voe03], we have the following properties of the motivic Steenrod squares: firstly, $\mathsf{Sq}^2(\tau) = 0$ since $\mathsf{Sq}^{2n}(u) = 0$ for any u of bidegree (p, q) with $n > p - q$ and $n \geq q$, and then $\mathsf{Sq}^3(\tau) = \mathsf{Sq}^1\mathsf{Sq}^2(\tau) = 0$; secondly, $\mathsf{Sq}^1 = \beta$ and $\beta\tau = \rho$, and finally, we have the motivic Cartan formula

$$\mathsf{Sq}^{2i+1}(u \smile v) = \sum_{r=0}^{2i+1} \mathsf{Sq}^r(u) \smile \mathsf{Sq}^{2i+1-r}(v) + \rho \sum_{r=0}^{i-1} \mathsf{Sq}^{2r+1}(u) \smile \mathsf{Sq}^{2i-2r-1}(v).$$

Using these facts, we compute

$$\begin{aligned}\mathsf{Sq}^3(\tau^2) &= \mathsf{Sq}^0(\tau)\mathsf{Sq}^3(\tau) + \mathsf{Sq}^1(\tau)\mathsf{Sq}^2(\tau) + \mathsf{Sq}^2(\tau)\mathsf{Sq}^1(\tau) + \mathsf{Sq}^3(\tau)\mathsf{Sq}^0(\tau) + \rho\mathsf{Sq}^1(\tau)\mathsf{Sq}^1(\tau) \\ &= \rho(\beta\tau)^2 = \rho^3,\end{aligned}$$

which proves Claim (2). □

Remark 3.10. The long exact sequence argument in the proof of Lemma 3.9 could have been replaced by a spectral sequence argument using the real realization of the very effective slice filtration on kgl and its multiplicative structure, as we will do later in the proof of Lemma 4.9 for ko instead. Then, computing the differential $d^1 : E^1_{2,0} = \pi_2(r_\mathbb{R}\tilde{s}_0\mathsf{kgl}) \to \pi_1(r_\mathbb{R}\tilde{s}_1\mathsf{kgl}) = E^1_{1,1}$ boils down to our proof of Lemma 3.9 (actually $d^1 = (\pi_2 \circ r_\mathbb{R})(d)$ where $d : \mathsf{H}\mathbb{Z} \to \Sigma^{3,1}\mathsf{H}\mathbb{Z}$ is the connecting map in the proof of the latter).

3.3. The real realization of $\widetilde{\mathsf{H}\mathbb{Z}}$.

The motivic spectrum $\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}_*^{MW}$ appears in the computation of the very effective slices of ko, which we will need for the proof of our main result about $r_\mathbb{R}(\mathsf{ko})$ in Section 4. It is endowed with the structure of a motivic $\mathcal{E}_\infty$-ring in the same way as $\mathsf{H}\mathbb{Z}$ and $\mathsf{H}\mathbb{Z}/2$ in Remark 3.2.

Proposition 3.11. *On the homotopy rings, the real realization of the quotient map*

$$\widetilde{\mathsf{H}\mathbb{Z}} := f_0\underline{K}_*^{MW} \longrightarrow f_0(\underline{K}_*^{MW}/\eta) = f_0\underline{K}_*^M = \mathsf{H}\mathbb{Z}$$

identifies with the quotient map $\mathbb{Z}[t^2]/(2t^2) \to \mathbb{Z}[t^2]/2$.

Proof. The homotopy ring of $r_\mathbb{R}\mathsf{H}\mathbb{Z}$ was computed in Proposition 3.6. By [Bac17, Lem. 17], the quotient map in the statement induces isomorphisms on homotopy sheaves $\underline{\pi}_n(-)_*$ for all $n \neq 0$. In particular, for all $n \neq 0$, $\pi_n(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) \cong \mathsf{colim}_m\underline{\pi}_n(\widetilde{\mathsf{H}\mathbb{Z}})_m(\mathbb{R}) \cong \mathsf{colim}_m\underline{\pi}_n(\mathsf{H}\mathbb{Z})_m(\mathbb{R}) \cong \pi_n(r_\mathbb{R}\mathsf{H}\mathbb{Z})$. Since $\widetilde{\mathsf{H}\mathbb{Z}}$ is very effective by Proposition 3.3, its real realization is connective (Lemma 1.11), and so we only have to deal with the zeroth homotopy group. Consider the unit map $\mathbb{1} \to \widetilde{\mathsf{H}\mathbb{Z}}$. By [Bac17, Lem. 17] and [Mor99, after Thm. 6.4.7], this map induces isomorphisms on the homotopy sheaves $\underline{\pi}_0(-)_*$, which agree with those of the sphere spectrum. In particular, arguing as above, $\pi_0(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) \cong \pi_0(r_\mathbb{R}\underline{K}_*^{MW}) \cong \pi_0(r_\mathbb{R}\mathcal{S}) = \pi_0(\mathbb{S}) = \mathbb{Z}$. We therefore have in degree 0 a ring map $\mathbb{Z} \to \mathbb{Z}/2$, which must be the quotient by 2 map. This suffices to conclude our proof, because we have determined $\pi_*(r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}) \to \pi_*(r_\mathbb{R}\mathsf{H}\mathbb{Z})$ in every degree, we know the ring structure on the right hand-side, and we know that the map is a ring homomorphism. □

3.4. **The real realization of KO.**

We now compute the real realization of KO (as defined in [BH20, §3.2] for example) as an $\mathcal{E}_\infty$-ring. The result was already proven at the level of spectra (see [BH20, Lem. 3.9]); the proof here is very similar, with more care to the multiplicative structure. In the proof, we will encounter the motivic $\mathcal{E}_\infty$-ring $\mathsf{KW} := \mathsf{KO}[\eta^{-1}]$ representing Balmer–Witt K-theory, see for example [Hor05, Appendix A].

Proposition 3.12. *There is an $\mathcal{E}_\infty$-map of spectra*

$$\gamma : \mathsf{L}(\mathbb{R}) \longrightarrow r_{\mathbb{R}}\mathsf{KW},$$

where L *denotes the L-theory spectrum, inducing an equivalence of $\mathcal{E}_\infty$-rings $\mathsf{L}(\mathbb{R})[1/2] \simeq r_{\mathbb{R}}\mathsf{KW}$. In particular, there is an equivalence of $\mathcal{E}_\infty$-rings*

$$r_{\mathbb{R}}\mathsf{KO} \simeq r_{\mathbb{R}}\mathsf{KO}[1/2] \simeq r_{\mathbb{R}}\mathsf{KW} \simeq \mathsf{KO}^{\mathsf{top}}[1/2].$$

Proof. By [CHN25, Cor. 8.1.8], we have $\Gamma(\mathbb{R}, \mathsf{KW}) \simeq \mathsf{L}(\mathbb{R})$ as $\mathcal{E}_\infty$-rings, where $\Gamma(\mathbb{R}, -)$ is the global sections functor, if we view a motivic spectrum as a presheaf of topological spectra (see for example [CHR24]). There is a slight technicality, in that we have to check that the $\mathcal{E}_\infty$-ring $\mathsf{KW}_{\mathbb{R}}$ from [CHN25, Def. 8.1.1] is indeed the same as the one we were previously considering. By [KR24], the motivic $\mathcal{E}_\infty$-ring $\mathsf{KQ}_{\mathbb{R}}$ constructed in [CHN25, Def. 8.1.1] is equivalent to the $\mathcal{E}_\infty$-ring KO we are considering. Therefore we want to show that the map of $\mathsf{KQ}_{\mathbb{R}}$-modules $w_{\mathbb{R}} : \mathsf{KQ}_{\mathbb{R}} \to \mathsf{KW}_{\mathbb{R}}$ from [CHN25, below Cor. 8.1.8] is indeed given by η-localization, which can be described as the natural map to the colimit of the telescope for multiplication by η. The map $w_{\mathbb{R}}$ is obtained by applying the lax symmetric monoidal, colimit-preserving functor $\mathcal{M}^s_{\mathbb{R}}$ [CHN25, Lem. 7.4.10] to the map of $\mathbb{GW}$-modules $\mathbb{GW} \to \mathbb{L}$ between stabilized Grothendieck-Witt theory and stabilized L-theory. By [Sch25, §3], the latter is also given by η'-localization, where η' is the image of the unit in $GW_0^{[0]}(\mathbb{Z})$ by the boundary map from [CHN25, Rmk. 8.1.11]. It is shown there that after applying the functor $\mathcal{M}^s_{\mathbb{R}}$, this boundary map is the usual η, namely the motivic Hopf map. The desired conclusion follows.

Then, since $\mathsf{KW}[1/\rho] \simeq \mathsf{KW}[1/2]$ (see the proof of Proposition 3.12), by Theorem 1.13, we have $r_{\mathbb{R}}\mathsf{KW} \simeq \Gamma(\mathbb{R}, \mathsf{KW}[1/\rho]) \simeq \Gamma(\mathbb{R}, \mathsf{KW}[1/2]) \simeq \Gamma(\mathbb{R}, \mathsf{KW})[1/2] \simeq \mathsf{L}(\mathbb{R})[1/2]$. By [LNS23, Ex. 9.2], there is an $\mathcal{E}_\infty$-map $\tau_{\mathbb{R}} : \mathsf{ko}^{\mathsf{top}} \to L(\mathbb{R})_{\geq 0}$, inducing an equivalence after inverting 2. Moreover, on the eighth homotopy groups, this maps sends λ_8 (a generator for $\pi_8(\mathsf{ko}^{\mathsf{top}}) \cong \mathbb{Z}$) to $16b^2$ (where b is a generator of $\pi_4(\mathsf{L}(\mathbb{R})) \cong \mathbb{Z}$). Therefore, there is an equivalence

$$\mathsf{ko}^{\mathsf{top}}[1/2][\lambda_8^{-1}] \simeq \mathsf{L}(\mathbb{R})_{\geq 0}[1/2][(16b^2)^{-1}] \simeq \mathsf{L}(\mathbb{R})_{\geq 0}[1/2][b^{-1}].$$

The left-hand side is equivalent to $\mathsf{KO}^{\mathsf{top}}[1/2]$ since $\mathsf{ko}^{\mathsf{top}}[\lambda_8^{-1}] \simeq \mathsf{KO}^{\mathsf{top}}$ by periodicity (see for example [LN18, Cor. 5.1]), and similarly the right-hand side is $\mathsf{L}(\mathbb{R})[1/2]$ because $\mathsf{L}(\mathbb{R})_{\geq 0}[b^{-1}] \simeq \mathsf{L}(\mathbb{R})$ (same reference).

This proves the last equivalence in the statement. To prove the first equivalence, consider the Wood cofiber sequence $\Sigma^{1,1}\mathsf{KO} \xrightarrow{\eta} \mathsf{KO} \longrightarrow \mathsf{KGL}$ [RØ16, Thm. 3.4]. In Milnor–Witt K-theory, we have $h\rho^2 = 0$. Indeed, by [Mor12, Cor. 3.8], Milnor–Witt K-theory is "$(1-h)$-graded commutative", so $\rho^2 = (1-h)\rho^2$. Therefore, after inverting ρ, or equivalently, after real realization (Theorem 1.13), we have $0 = h = 2+\rho\eta$. Then the real realization of η is equivalent to multiplication by -2. But also, we have seen in Lemma 3.7 that $r_{\mathbb{R}}\mathsf{KGL} \simeq 0$. This means that the cofiber of the multiplication by 2 map on $r_{\mathbb{R}}\mathsf{KO}$ is zero, in other terms $\cdot 2$ is an equivalence, and thus $r_{\mathbb{R}}\mathsf{KO} \simeq r_{\mathbb{R}}(\mathsf{KO}[1/2])$ via the canonical localization map. Also, the above implies that $r_{\mathbb{R}}(\mathsf{KO}[1/2])$ identifies with $\mathsf{KO}[\rho^{-1}, 1/2] \simeq \mathsf{KO}[\rho^{-1}, \eta^{-1}] \simeq \mathsf{KW}[\rho^{-1}]$ under the equivalence $\mathsf{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathsf{Sp}$. Then $r_{\mathbb{R}}(\mathsf{KO}[1/2]) \simeq r_{\mathbb{R}}(\mathsf{KW})$. This establishes the first and second equivalences in the statement. □

Let $\mathsf{kw} := \mathsf{KW}_{\geq 0}$ be the connective cover of KW in the homotopy t-structure. It inherits the structure of a motivic $\mathcal{E}_\infty$-ring (Remark 1.3). Also let $\mathsf{ko}^{\mathsf{top}} := \mathsf{KO}^{\mathsf{top}}_{\geq 0}$ be the connective cover of the real topological K-theory spectrum.

Proposition 3.13. *The equivalences $r_{\mathbb{R}}\mathsf{KO}[1/2] \simeq r_{\mathbb{R}}\mathsf{KW} \simeq \mathsf{KO}^{\mathsf{top}}[1/2]$ of Proposition 3.12 induce equivalences of $\mathcal{E}_\infty$-rings*

$$\mathsf{ko}^{\mathsf{top}}[1/2] \simeq r_{\mathbb{R}}\mathsf{kw} \simeq r_{\mathbb{R}}\mathsf{ko}[1/2].$$

Proof. By [BH20, Lem. 6.9], the composition $\mathsf{ko} \to \mathsf{KO} \to \mathsf{KO}[\eta^{-1}] =: \mathsf{KW}$ induces an equivalence $\mathsf{ko}[\eta^{-1}] \simeq \mathsf{KW}_{\geq 0} =: \mathsf{kw}$. Then, as in the proof of Proposition 3.12, localization away from η and away from 2 induce equivalences $r_{\mathbb{R}}\mathsf{kw} \simeq r_{\mathbb{R}}\mathsf{kw}[1/2] \simeq r_{\mathbb{R}}\mathsf{ko}[1/2]$. Note that the functor $r_{\mathbb{R}}$ is t-exact: by Definition 1.2, $E \in \mathsf{SH}(k)$ belongs to $\mathsf{SH}(k)_{\geq 0}$ if and only if $\underline{\pi}_i(E)_* = 0$ for all $i < 0$ (respectively, $\mathsf{SH}(k)_{\leq 0}$ and $i > 0$), and by Lemma 1.14 we have $\pi_i(r_{\mathbb{R}}(E)) \cong \mathsf{colim}_n \underline{\pi}_i(E)_n(\mathbb{R})$ for all $i \in \mathbb{Z}$. Therefore, $r_{\mathbb{R}}\mathsf{kw} \simeq r_{\mathbb{R}}(\mathsf{KW})_{\geq 0} \simeq \mathsf{KO}^{\mathsf{top}}[1/2]_{\geq 0} \simeq \mathsf{ko}^{\mathsf{top}}[1/2]$ by Proposition 3.12. □

4. The real realization of ko

The motivic spectrum ko inherits an $\mathcal{E}_\infty$-ring structure from that of KO, because the very effective cover functor is lax symmetric monoidal by Proposition 1.8. Thus, $r_\mathbb{R}\mathsf{ko}$ is also an $\mathcal{E}_\infty$-ring. In this section, we identify the underlying $\mathcal{E}_1$-structure by an explicit 2-local fracture square. In Subsection 4.1, we will compute the homotopy ring of $r_\mathbb{R}\mathsf{ko}$, thanks to the Wood cofiber sequence $\Sigma^{1,1}\mathsf{ko} \to \mathsf{ko} \to \mathsf{kgl}$ [ARØ20, Prop. 2.11] and the knowledge of the very effective slices of ko (computed in [Bac17], see Theorem 4.2 below), which involves only motivic spectra whose real realizations we have already computed in Section 3. This homotopy ring is a polynomial ring over $\mathbb{Z}$ with a single generator in degree 4.

But this is not enough to identify $r_\mathbb{R}\mathsf{ko}$ as an $\mathcal{E}_1$-ring. Indeed, by Proposition 1.19, the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}$-algebra $\mathsf{H}\mathbb{Z}[t^4]$ has the same homotopy ring, but they are not equivalent (see Remark 4.4). Inverting 2 makes things easier; we have already identified $r_\mathbb{R}(\mathsf{ko})[1/2]$ in Proposition 3.13. This might be a good reason to consider a 2-local fracture square, expressing $r_\mathbb{R}\mathsf{ko}$ as the pullback of $r_\mathbb{R}(\mathsf{ko})[1/2]$ and $r_\mathbb{R}(\mathsf{ko})_{(2)}$ over $r_\mathbb{R}(\mathsf{ko})_\mathbb{Q}$. Our approach is to identify these terms and the maps in the square more explicitly, taking into account the $\mathcal{E}_1$-structures. This will use the description of free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras and their homotopy rings from Proposition 1.21; this is one reason for which we identify the $\mathcal{E}_1$-structure only (and not the $\mathcal{E}_\infty$-one). The resulting 2-local fracture square coincides with the one obtained in [HLN21] for the connective L-theory spectrum of $\mathbb{R}$. As a corollary, we get an equivalence of $\mathcal{E}_1$-rings $r_\mathbb{R}(\mathsf{ko}) \simeq \mathsf{L}(\mathbb{R})_{\geq 0}$.

4.1. The homotopy ring of $r_\mathbb{R}(\mathsf{ko})$.

Proposition 4.1. *The homotopy ring of $r_\mathbb{R}\mathsf{ko}$ is the polynomial ring $\pi_*(r_\mathbb{R}\mathsf{ko}) \cong \mathbb{Z}[x]$, where x is a generator in degree 4, mapping to a generator of $\pi_4(r_\mathbb{R}\mathsf{kgl}) \cong \mathbb{Z}/2$ (see Proposition 3.8) in the real realization of the Wood cofiber sequence $\Sigma^{1,1}\mathsf{ko} \to \mathsf{ko} \to \mathsf{kgl}$.*

Proof. We will proceed in four steps:

(1) We first show that these homotopy groups are finitely generated in every degree, using the description of the very effective slices (Definition 1.7) of ko.
(2) Using the Wood cofiber sequence, we show that these groups have no 2-torsion.
(3) We then show that they have no torsion at the other primes, and that they are actually free abelian groups of rank 1 in every non-negative degree divisible by 4, and 0 else. To do so, we reduce to a statement after inverting 2, and use the computation of $r_\mathbb{R}\mathsf{ko}[1/2]$ in Proposition 3.13.
(4) Finally, we describe the ring structure, studying again the Wood cofiber sequence.

Step 1: *The homotopy groups of $r_\mathbb{R}\mathsf{ko}$ are finitely generated abelian groups.* We claim that: ($\star$) the real realizations of the very effective slices of ko have degreewise finitely generated homotopy groups. Then, we show by induction on m the following statement: for all $n \in \mathbb{N}$, $\pi_{n+m}(r_\mathbb{R}(\tilde{f}_n\mathsf{KO}))$ is finitely generated. This holds for $m \leq -1$ since real realization on $\mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(n)$ takes values in n-connective spectra as seen in Lemma 1.11. This implies our result because for $n = 0$, we get that for all $m \in \mathbb{N}$, $\pi_m(r_\mathbb{R}(\tilde{f}_0\mathsf{KO})) = \pi_m(r_\mathbb{R}\mathsf{ko})$ is finitely generated. Assume that the induction hypothesis holds for some fixed $m \geq -1$. Then for all $n \in \mathbb{N}$, the cofiber sequence $\tilde{f}_{n+1}\mathsf{KO} \to \tilde{f}_n\mathsf{KO} \to \tilde{s}_n\mathsf{KO}$ defining the very effective slices induces a long exact sequence

$$\cdots \xrightarrow{a} \pi_{n+1+m}(r_\mathbb{R}(\tilde{f}_{n+1}\mathsf{KO})) \longrightarrow \pi_{n+m+1}(r_\mathbb{R}(\tilde{f}_n\mathsf{KO})) \longrightarrow \pi_{n+m+1}(r_\mathbb{R}(\tilde{s}_n\mathsf{KO}))$$
$$\xrightarrow{\quad b \quad}$$
$$\pi_{n+m}(r_\mathbb{R}(\tilde{f}_{n+1}\mathsf{KO})) \longrightarrow \pi_{n+m}(r_\mathbb{R}(\tilde{f}_n\mathsf{KO})) \longrightarrow \cdots$$

and therefore $\pi_{n+m+1}(r_\mathbb{R}(\tilde{f}_n\mathsf{KO}))$ is an extension of $\mathsf{ker}(b)$ and $\mathsf{coker}(a)$. The former is finitely generated because it is a subgroup of $\pi_{n+m+1}(r_\mathbb{R}(\tilde{s}_n\mathsf{KO}))$, which is finitely generated by ($\star$). The latter is also finitely generated since it is a quotient of $\pi_{n+1+m}(r_\mathbb{R}(\tilde{f}_{n+1}\mathsf{KO}))$, which is finitely generated by the induction hypothesis. Therefore, $\pi_{n+m+1}(r_\mathbb{R}(\tilde{f}_n\mathsf{KO}))$ is finitely generated, finishing our proof by induction. To finish Step 1, it remains to prove ($\star$).

Theorem 4.2 ([Bac17, Thm. 16])**.** *The very effective slices of KO are given by $\tilde{s}_n\mathsf{KO} \simeq \Sigma^{2n,n}\tilde{S}_{n \bmod 4}$ where*

$$\tilde{S}_0 = \tilde{s}_0\mathsf{KO}, \quad \tilde{S}_1 = \mathsf{H}\mathbb{Z}/2, \quad \tilde{S}_2 = \mathsf{H}\mathbb{Z}, \quad \tilde{S}_3 = 0,$$

and there is a cofiber sequence

$$\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \longrightarrow \tilde{s}_0\mathsf{KO} \longrightarrow \widetilde{\mathsf{H}\mathbb{Z}}.$$

By the computation of the real realizations of $\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$ and $\widetilde{\mathsf{H}\mathbb{Z}}$ in Subsections 3.1 and 3.3, we know that their homotopy groups are finitely generated in every degree. Thus, that of the realizations of very effective slices of ko are too (for $\tilde{s}_0$, use the same inductive argument as we just did).

Step 2: *There is no 2-torsion in* $\pi_*(r_{\mathbb{R}}\mathsf{ko})$. The Wood cofiber sequence seen in the proof of Proposition 3.12 refines to one on the very effective covers: by [ARØ20, Prop. 2.11], there is a cofiber sequence $\Sigma^{1,1}\mathsf{ko} \xrightarrow{\eta} \mathsf{ko} \xrightarrow{c} \mathsf{kgl}$, where η is induced by the Hopf map and c is induced by the forgetful map from Hermitian K-theory to algebraic K-theory. As we computed in the case of KO (Proposition 3.12), the real realization of η is the multiplication by 2 map on the sphere spectrum, and by Proposition 3.7, the $\mathcal{E}_1$-ring $r_{\mathbb{R}}\mathsf{kgl}$ is the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}/2$-algebra $\mathsf{H}\mathbb{Z}/2[t^4]$. Assume for a contradiction that $\pi_m(r_{\mathbb{R}}\mathsf{ko})$ has 2-torsion for some $m \geq 0$. Then, in the long exact sequence induced by the Wood cofiber sequence

$$\cdots \longrightarrow \pi_{m+1}(r_{\mathbb{R}}\mathsf{kgl}) \xrightarrow{a} \pi_m(r_{\mathbb{R}}\mathsf{ko}) \xrightarrow{\cdot 2 = b} \pi_m(r_{\mathbb{R}}\mathsf{ko}) \xrightarrow{c} \pi_m(r_{\mathbb{R}}\mathsf{kgl}) \longrightarrow \cdots,$$

the kernel of the map $\cdot 2 = b$ is non-trivial, so a is non-zero and $\pi_{m+1}(r_{\mathbb{R}}\mathsf{kgl}) \neq 0$, i.e., $k \equiv 3 \mod 4$. Then $\pi_m(r_{\mathbb{R}}\mathsf{kgl}) = 0$, meaning that multiplication by 2 is surjective on $\pi_m(r_{\mathbb{R}}\mathsf{ko})$. In Step 1, we saw that this group was finitely generated abelian. Using the classification of such groups, we see that this is impossible for non-zero groups with 2-torsion.

Step 3: *The groups* $\pi_m(r_{\mathbb{R}}\mathsf{ko})$ *are torsion-free of rank 1 in degrees* $m \in 4\mathbb{N}$*, and zero otherwise.* By Proposition 3.13 and since real realization and homotopy groups commutes with localization away from 2 (Lemma 1.15), we have isomorphisms of graded abelian groups

$$\begin{aligned}\pi_*(r_{\mathbb{R}}\mathsf{ko})[1/2] &\cong \pi_*((r_{\mathbb{R}}\mathsf{ko})[1/2]) \cong \pi_*(r_{\mathbb{R}}(\mathsf{ko}[1/2])) \cong \pi_*(\mathsf{ko}^{\mathsf{top}}[1/2]) \\ &\cong \left(\mathbb{Z}[\alpha_1, \beta_4, \lambda_8] \Big/ (\alpha_1^3, 2\alpha_1, \alpha_1\beta_4, \beta_4^2 - 4\lambda_8)\right)[1/2] \cong \mathbb{Z}[1/2][\beta_4]\end{aligned}$$

(where $|\alpha_1| = 1$, $|\beta_4| = 4$, and $|\lambda_8| = 8$) by a classical result in topology, see for instance [Str92, §1.1]. In particular, the homotopy groups of $r_{\mathbb{R}}\mathsf{ko}$ cannot have any p-torsion for $p \neq 2$ either. Otherwise, their localizations away from 2 would still have torsion, but $\mathbb{Z}[1/2]$ does not. Therefore, $\pi_*(r_{\mathbb{R}}\mathsf{ko})$ is degreewise free. Comparing with the localization away from 2, we deduce that it consists in a copy of $\mathbb{Z}$ in every degree divisible by 4, and the trivial group else. Indeed, if $\mathbb{Z}^{\oplus m}[1/2] \cong \mathbb{Z}[1/2]$ for some $m \in \mathbb{N}$, then the quotients by the $\mathbb{Z}$-module map $\cdot 3$ on both sides are isomorphic as well. They are respectively $(\mathbb{Z}/3)^{\oplus m}$ and $\mathbb{Z}/3$, whence $m = 1$ for cardinality reasons.

Step 4: *We describe the ring structure.* The Wood cofiber sequence induced a long exact sequence

$$\cdots \longrightarrow \pi_{4m}(r_{\mathbb{R}}\mathsf{ko}) \xrightarrow{c_{4m}} \pi_{4m}(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2 \longrightarrow \pi_{4m-1}(r_{\mathbb{R}}\mathsf{ko}) = 0 \longrightarrow \cdots.$$

Let x be a generator of $\pi_4(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}$. Then since c_4 is surjective, $c_4(x)$ generates $\pi_4(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2$. The forgetful map $c : \mathsf{ko} \to \mathsf{kgl}$ in the Wood cofiber sequence is a map of $\mathcal{E}_\infty$-rings, and thus induces a ring homomorphism in homotopy. Therefore, for $m \geq 0$, $c_{4m}(x^m) = c_4(x)^m$ generates $\pi_{4m}(r_{\mathbb{R}}\mathsf{kgl}) \cong \mathbb{Z}/2$ as well (since $\pi_*(r_{\mathbb{R}}\mathsf{kgl})$ is a polynomial ring by Proposition 3.8). We claim that x^m generates $\pi_{4m}(r_{\mathbb{R}}\mathsf{ko})$. Indeed, write $x^m = a \cdot g_m$ with $a \in \mathbb{Z}$ and g_m is a generator of $\pi_{4m}(r_{\mathbb{R}}\mathsf{ko})$. By Proposition 3.13, we have ring isomorphisms $\pi_*(r_{\mathbb{R}}\mathsf{ko}) \otimes \mathbb{Z}[1/2] \cong \pi_*(\mathsf{ko}^{\mathsf{top}}[1/2]) \cong \mathbb{Z}[1/2][\beta_4]$. In degree 4, since x generates $\pi_4(r_{\mathbb{R}}\mathsf{ko})$, under this isomorphism x must be sent to a generator of $\mathbb{Z}[1/2]\{\beta_4\}$ as a $\mathbb{Z}[1/2]$-module, i.e., $c\beta_4$ with $c \in \mathbb{Z}[1/2]^\times$. In particular, x^m corresponds to a generator $c^m\beta_4^m$ of the degree $4m$ part. Therefore, there exists $b \in \mathbb{Z}[1/2]$ with $g_m = b \cdot x^m = ba \cdot g_m$, so $a \in \mathbb{Z}$ is a unit in $\mathbb{Z}[1/2]$. Thus, if $a \neq \pm 1$, then it must be even. In the latter case, we would have $c_{4m}(x^m) = a \cdot c_{4m}(g_m) = 0 \in \mathbb{Z}/2$, which is a contradiction. We conclude that $a = \pm 1$, whence our claim follows. □

4.2. **A 2-local fracture square for the $\mathcal{E}_1$-ring structure on $r_{\mathbb{R}}(\mathsf{ko})$.**

4.2.1. *Identifying the localization at* (2).

Since $\pi_*(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}[x]$ by Proposition 4.1, we have $\pi_*((r_{\mathbb{R}}\mathsf{ko})_{(2)}) \cong \pi_*(r_{\mathbb{R}}\mathsf{ko}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}[x]$. We want to use this to identify $(r_{\mathbb{R}}\mathsf{ko})_{(2)}$ with the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra on one generator in degree 4. Inspired by the computation of a 2-local fracture square for $\mathsf{L}(\mathbb{R})_{\geq 0}$ in [HLN21, p3], to prove this we use the spectrum MSL as an intermediate step. This is where Theorem 2.45 comes in handy, since we have shown that $r_{\mathbb{R}}(\mathsf{MSL}) \simeq \mathsf{MSO}$ as $\mathcal{E}_\infty$-rings.

Proposition 4.3. *There is an equivalence of $\mathcal{E}_1$-rings*

$$\mathsf{H}\mathbb{Z}_{(2)}[t^4] \xrightarrow{\simeq} (r_{\mathbb{R}}\mathsf{ko})_{(2)}$$

sending $t^4 \in \pi_4(\mathsf{H}\mathbb{Z}_{(2)}[t^4])$ to (the image in the localization of) the generator $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$.

Proof. By [HJNY22, Thm. 10.1], there is an $\mathcal{E}_\infty$-ring morphism $\mathsf{MSL} \to \mathsf{ko}$. Its real realization is an $\mathcal{E}_\infty$-map $\mathsf{MSO} \simeq r_{\mathbb{R}}(\mathsf{MSL}) \to r_{\mathbb{R}}\mathsf{ko}$ (Theorem 2.45). Now, by [HLN21, Cor. 3.7], MSO admits after localization at (2) a map of $\mathcal{E}_2$-rings $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{MSO}_{(2)}$. Therefore, by composition, we obtain an $\mathcal{E}_2$-map $\mathsf{H}\mathbb{Z}_{(2)} \to (r_{\mathbb{R}}\mathsf{ko})_{(2)}$. By Proposition 1.21(*iii*), to obtain a map of $\mathcal{E}_1$-rings $\mathsf{H}\mathbb{Z}_{(2)}[t] \to (r_{\mathbb{R}}\mathsf{ko})_{(2)}$, it then suffices to specify an element of $\pi_4((r_{\mathbb{R}}\mathsf{ko})_{(2)})$. We choose the image in the localization of $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$. By Proposition 1.21(*ii*), and since $\pi_*((r_{\mathbb{R}}\mathsf{ko})_{(2)}) \cong \pi_*(r_{\mathbb{R}}\mathsf{ko}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}[x]$, this map induces isomorphisms on the homotopy groups (note that $\mathbb{Z}_{(2)}$ is a module over itself in a unique way), whence the desired equivalence $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \xrightarrow{\simeq} (r_{\mathbb{R}}\mathsf{ko})_{(2)}$. □

Remark 4.4. Localization at (2) is crucial here, because MSO is not an $\mathsf{H}\mathbb{Z}$-module, since it has non-trivial k-invariants. The k-invariants of $E \in \mathsf{Sp}$ are the horizontal compositions in the diagram

$$\begin{array}{ccccc}
\Sigma^{n+2}\mathsf{H}(\pi_{n+2}E) & \longrightarrow & E_{\leq n+2} & & \\
 & & \downarrow & & \\
\Sigma^{n+1}\mathsf{H}(\pi_{n+1}E) & \longrightarrow & E_{\leq n+1} & \longrightarrow & \Sigma^{n+3}\mathsf{H}(\pi_{n+2}E) \\
 & & \downarrow & & \\
 & & E_{\leq n} & \longrightarrow & \Sigma^{n+2}\mathsf{H}(\pi_{n+1}E)
\end{array}$$

where the right-down-right zigzags (one showed in blue, the other one in green) are distinguished triangles, or the maps defined in the same way using the Whitehead tower instead of the Postnikov tower. In particular, the strategy above cannot be used to identify $r_{\mathbb{R}}\mathsf{ko}$ as an $\mathcal{E}_1$-ring with $\mathsf{H}\mathbb{Z}[t^4]$ directly. Actually, we can even prove that this is wrong. Otherwise, we would have $\mathsf{ko}^{\mathsf{top}}[1/2] \simeq r_{\mathbb{R}}\mathsf{ko}[1/2] \simeq \mathsf{H}\mathbb{Z}[1/2][t^4]$ as $\mathcal{E}_1$-rings. We would then have $\mathsf{ku}[1/2] \wedge \mathsf{H}\mathbb{Z} \simeq \mathsf{ko}^{\mathsf{top}}[1/2] \wedge \mathsf{H}\mathbb{Z} \oplus \Sigma^2\mathsf{ko}^{\mathsf{top}}[1/2] \wedge \mathsf{H}\mathbb{Z}$ (using the proof of [LNS23, Thm. 9.3]), and thus $\mathsf{ku}[1/2] \wedge \mathsf{H}\mathbb{Z} \simeq \mathsf{H}\mathbb{Z}[1/2][t^4] \wedge \mathsf{H}\mathbb{Z} \oplus \Sigma^2\mathsf{H}\mathbb{Z}[1/2][t^4] \wedge \mathsf{H}\mathbb{Z}$. Reducing $\mathsf{mod}\,3$, this yields $\mathsf{ku}[1/2] \wedge \mathsf{H}\mathbb{Z}/3 \simeq \mathsf{H}\mathbb{Z}/3[t^4] \wedge \mathsf{H}\mathbb{Z} \oplus \Sigma^2\mathsf{H}\mathbb{Z}/3[t^4] \wedge \mathsf{H}\mathbb{Z}$. Using formal group laws, one shows that the left-hand side is trivial, while the right-hand side is not (see for example [GS12]).

4.2.2. *Identifying the rationalization.*

Using our knowledge of the localization at (2) of $r_{\mathbb{R}}\mathsf{ko}$, we can identify its rationalization.

Proposition 4.5. *There is an equivalence of $\mathcal{E}_1$-rings*

$$\mathsf{H}\mathbb{Q}[t^4] \xrightarrow{\simeq} (r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}$$

sending $t^4 \in \pi_4(\mathsf{H}\mathbb{Q}[t^4])$ to (the image in the localization of) the generator $x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$.

Proof. By Proposition 4.3, we have $(r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}} \simeq (r_{\mathbb{R}}\mathsf{ko})_{(2)}[1/2] \cong \mathsf{H}\mathbb{Z}_{(2)}[t^4][1/2]$. We claim that the latter is equivalent to the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebra $\mathsf{H}\mathbb{Q}[t^4]$. Indeed, the unique ring map $\mathbb{Z}_{(2)} \to \mathbb{Q}$ (which induces an $\mathcal{E}_\infty$-map $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{H}\mathbb{Q}$ and thus a map of $\mathsf{H}\mathbb{Z}_{(2)}$-modules $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{H}\mathbb{Q}[t^4]$) and the element $t^4 \in \pi_4(\mathsf{H}\mathbb{Q}[t^4])$ induce a map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \to \mathsf{H}\mathbb{Q}[t^4]$. Comparing the homotopy rings (Proposition 1.21(*ii*)), we see that after localization away from 2, this map induces isomorphisms in homotopy, as desired. □

4.2.3. *Main result: identifying the maps and assembling the fracture square.*

As the title indicates, we now combine the results of the previous subsection to identify the advertised explicit 2-local fracture square for $r_{\mathbb{R}}\mathsf{ko}$. We first define the Chern character map; it will appear in our fracture square.

Definition 4.6. Let $\mathsf{H}\mathbb{Q}[t^4]$ and $\mathsf{H}\mathbb{Q}[u^2]$ be the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Q}$-algebras on one generator in degrees 4 and 2 respectively. We call *Chern character* any of the following:

- the map $\mathsf{ku} := \mathsf{KU}_{\geq 0} \to \mathsf{H}\mathbb{Q}[u^2]$ given by the (connective cover of the) classical Chern character as defined in [BH58, §9.1]. In homotopy, it induces the inclusion $\mathbb{Z}[u^2] \to \mathbb{Q}[u^2]$ where $u^2 \in \pi_2(\mathbb{Z} \times BU) \cong \mathbb{Z}$ is a generator.

- the map $\mathsf{ko}^{\mathsf{top}} \to \mathsf{H}\mathbb{Q}[t^4]$ obtained by pre-composition of the previous map with the connective cover the complexification map $\mathsf{KO}^{\mathsf{top}} \to \mathsf{KU}$ (induced by complexification of vector bundles) (their composition factors through the inclusion $\mathsf{H}\mathbb{Q}[t^4] \hookrightarrow \mathsf{H}\mathbb{Q}[u^2]$ given by $t^4 \mapsto (u^2)^2$). In homotopy, it induces the map $\mathbb{Z}[\alpha_1, \beta_4, \lambda_8]/(\alpha_1^3, 2\alpha_1, \alpha_1\beta_4, \beta_4^2 - 4\lambda_8) \to \mathbb{Z}[u^2]$ sending α_1 to 0, β_4 to $2(u^2)^2$, and thus λ_8 to $(u^2)^4$ (see for example [Rog08, §5.3]).
- the maps induced by the two previous ones after localization away from 2 of the source.

Remark 4.7. The Chern character $\mathsf{ch} : \mathsf{ko}^{\mathsf{top}} \to \mathsf{H}\mathbb{Q}[t^4]$ factors through the rationalization, and the map $\mathsf{ch}_{\mathbb{Q}} : \mathsf{ko}^{\mathsf{top}}_{\mathbb{Q}} \to \mathsf{H}\mathbb{Q}[t^4]$ obtained in this way is an equivalence. Indeed, $\pi_*(\mathsf{ko}^{\mathsf{top}}_{\mathbb{Q}}) \cong \pi_*(\mathsf{ko}^{\mathsf{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[\beta_4]$ and $\pi_*(\mathsf{H}\mathbb{Q}[t^4]) \cong \mathbb{Q}[t^4]$ by Proposition 1.21(*ii*), with $\mathsf{ch}_{\mathbb{Q}}$ sending β_4 to $2t^4$: it induces isomorphisms in homotopy.

Theorem 4.8. *There is a Cartesian square of $\mathcal{E}_1$-rings (and thus also of spectra)*

$$\begin{array}{ccc} r_{\mathbb{R}}\mathsf{ko} & \xrightarrow{x \mapsto \beta_4/2} & \mathsf{ko}^{\mathsf{top}}[1/2] \\ {\scriptstyle x \mapsto t^4}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{ch}\ \beta_4/2 \mapsto t^4} \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] & \xrightarrow[t^4 \mapsto t^4]{} & \mathsf{H}\mathbb{Q}[t^4]. \end{array}$$

where $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$ is the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra on a generator of degree 4 (Definition 1.20), and similarly for $\mathsf{H}\mathbb{Q}[t^4]$; and the map ch is the Chern character *from Definition 4.6. The assignments labeling the arrows describe the maps induced on the fourth homotopy groups.*

Proof. As mentioned in the introduction to this section, the 2-local fracture square of $r_{\mathbb{R}}\mathsf{ko}$ reads as the pullback square of $\mathcal{E}_1$-rings

$$\begin{array}{ccc} r_{\mathbb{R}}\mathsf{ko} & \longrightarrow & (r_{\mathbb{R}}\mathsf{ko})[1/2] \\ \downarrow & \lrcorner & \downarrow \\ (r_{\mathbb{R}}\mathsf{ko})_{(2)} & \longrightarrow & (r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}. \end{array}$$

This holds in general for $r_{\mathbb{R}}\mathsf{ko}$ replaced with any spectrum or $\mathcal{E}_n$-ring for $1 \leq n \leq \infty$, and 2 replaced by any prime p (to check that such a square is a pullback, it suffices to show that it is the case after inverting p, and also after moding out p. In both cases, the squares are trivial pullbacks because two parallel sides are identity maps).

Using the descriptions of the $\mathcal{E}_1$-rings $(r_{\mathbb{R}}\mathsf{ko})[1/2]$, $(r_{\mathbb{R}}\mathsf{ko})_{(2)}$, and $(r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}}$ in Propositions 3.13, 4.3, and 4.5 respectively, we obtain a commutative diagram

$$\begin{array}{ccccccc} r_{\mathbb{R}}\mathsf{ko} & \longrightarrow & (r_{\mathbb{R}}\mathsf{ko})[1/2] & \xrightarrow[g]{\simeq} & \mathsf{ko}^{\mathsf{top}}[1/2] & & \\ \downarrow & & \downarrow & & \downarrow & \searrow^{\mathsf{ch}} & \\ (r_{\mathbb{R}}\mathsf{ko})_{(2)} & \longrightarrow & (r_{\mathbb{R}}\mathsf{ko})_{\mathbb{Q}} & \xrightarrow[g_{\mathbb{Q}}]{\simeq} & (\mathsf{ko}^{\mathsf{top}}[1/2])_{\mathbb{Q}} & \xrightarrow[\mathsf{ch}_{\mathbb{Q}}]{\simeq} & \mathsf{H}\mathbb{Q}[t^4] \\ {\scriptstyle f}\uparrow{\scriptstyle \simeq} & & & & & & \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] & & \multicolumn{3}{c}{\text{- - - - - - - - - - - - - - - - - - -}\ {\scriptstyle k}} & & \end{array}$$

where f is the equivalence of Proposition 4.3, g is the equivalence of Proposition 3.13, $\mathsf{ch}_{\mathbb{Q}}$ is the map from Remark 4.7, and k is chosen so that the diagram commutes. In particular, the outer quadrilateral provides a pullback square

$$\begin{array}{ccc} r_{\mathbb{R}}\mathsf{ko} & \xrightarrow{x \mapsto g(x)} & \mathsf{ko}^{\mathsf{top}}[1/2] \\ {\scriptstyle x \mapsto t^4}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{ch}} \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] & \xrightarrow[t^4 \mapsto k(t^4)]{} & \mathsf{H}\mathbb{Q}[t^4]. \end{array}$$

We thus have to compute $k(t^4) = \mathsf{ch}(g(x))$. Looking back at the proof of Proposition 3.13, g is the inverse of the map g' induced on the connective covers by the top row in the diagram

$$\begin{array}{ccccccc} \mathsf{KO}^{\mathsf{top}}[1/2] & \xrightarrow{\tau_{\mathbb{R}}} & \mathsf{L}(\mathbb{R})[1/2] \simeq \Gamma(\mathbb{R}, \mathsf{KW}[1/2]) & \longrightarrow & \Gamma(\mathbb{R}, \mathsf{KW}[1/\rho, 1/2]) \simeq r_{\mathbb{R}}\mathsf{KO}[1/2] \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle \simeq} \\ \mathsf{KO}^{\mathsf{top}} & \xrightarrow{\tau_{\mathbb{R}}} & \mathsf{L}(\mathbb{R}) \simeq \Gamma(\mathbb{R}, \mathsf{KW}) & \longrightarrow & \Gamma(\mathbb{R}, \mathsf{KW}[1/\rho]) \simeq r_{\mathbb{R}}\mathsf{KW} \end{array}$$

where all maps in the square on the right-hand side are induced by the localizations (away from 2 or ρ), and the map $\tau_{\mathbb{R}}$ is defined in [LNS23, Thm. A and Ex. 9.2], where it is also proven that it sends $\beta_4 \in \pi_4(\mathsf{KO}^{\mathsf{top}})$ to $8b \in \pi_4(\mathsf{L}(\mathbb{R})) \simeq \mathbb{Z}$. We claim that $b = \pm\eta^{-4}\beta_{\mathsf{KO}} \in \pi_4(\Gamma(\mathbb{R}, \mathsf{KW})) \cong \pi_4(\mathsf{L}(\mathbb{R}))$. Indeed, $\mathsf{KW} = \mathsf{KO}[\eta^{-1}]$ is β_{KO}- and η-periodic, and thus multiplication by $\eta^{-4}\beta_{\mathsf{KO}}$ is an equivalence $\Sigma^4\mathsf{KW} \xrightarrow{\simeq} \mathsf{KW}$, and induces an isomorphism

$$\mathbb{Z} \cong \pi_0(\Gamma(\mathbb{R}, \mathsf{KW})) \cong [\mathcal{S}^4, \Sigma^4\mathsf{KW}] \xrightarrow{\cong} [\mathcal{S}^4, \mathsf{KW}] \cong \pi_4(\Gamma(\mathbb{R}, \mathsf{KW})) \cong \mathbb{Z}\{b\}.$$

So $\eta^{-4}\beta_{\mathsf{KO}} = \pm b$. Using the fact that $r_{\mathbb{R}}(\eta) = -2$ (see the proof of Proposition 3.12), it follows that $r_{\mathbb{R}}(b) = \pm r_{\mathbb{R}}(\beta_{\mathsf{KO}})/16 \in \pi_4(\Gamma(\mathbb{R}, \mathsf{KW}[1/\rho]))$. We defer to Proposition 4.9 below the proof that $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = \pm 16x$. It follows that $g'(\beta_4) = \pm r_{\mathbb{R}}(8b) = \pm 8x$, and thus $g(x) = \pm\beta_4/8$. Then $k(x) = \mathsf{ch}(g(x)) = \pm t^4/4$ (see Remark 4.7).

In order to be able to compare our fracture square with the one in [HLN21, p3] (Theorem 4.17), let us replace $g : r_{\mathbb{R}}\mathsf{ko}[1/2] \to \mathsf{ko}^{\mathsf{top}}[1/2]$ by $g'' := \psi^2 \circ g$, where $\psi^2 : \mathsf{ko}^{\mathsf{top}}[1/2] \to \mathsf{ko}^{\mathsf{top}}[1/2]$ is the second Adams operation. By [Kar78, Section IV, 7.13, 7.19 and 7.25], the latter is an $\mathcal{E}_\infty$-map, and it induces multiplication by 4 on the fourth homotopy groups. Then, repeating the same proof, we obtain $g''(x) = \psi^2(\pm\beta_4/8) = \pm\beta_4/2$ and $k(t^4) = \pm t^4$. We can further replace g'' by $-g''$ if needed to obtain $g''(x) = \beta_4/2$ and $k(t^4) = t^4$. Thus, the lower horizontal map k in our pullback square is the map of $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebras $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \to \mathsf{H}\mathbb{Q}[t^4]$ sending t^4 to t^4 in the fourth homotopy groups, as desired. □

Proposition 4.9. *In the notation of Proposition 4.1, we have $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = \pm 16x \in \pi_4(r_{\mathbb{R}}\mathsf{ko})$.*

Proof. The proof is divided in several lemmas proven in Subsection 4.2.5. We explain here how they combine to prove the proposition.

Step 1: *Reducing the problem to computing the cardinality of a certain homotopy group.* We want to understand the action of $r_{\mathbb{R}}\beta_{\mathsf{ko}}$ on the homotopy groups of $r_{\mathbb{R}}\mathsf{ko}$, by considering a cofiber sequence where this map appears together with only a small number of easily computable groups. We will thus consider slices. By Lemma 4.12 below, if we write $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = ax$ with $a \in \mathbb{Z}$ (recall that x generates $\pi_4(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}$), then $a = \pm|\pi_4(r_{\mathbb{R}}C)|$, where C is the cofiber of the natural map $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$. This result holds because, as we will see in its proof, the natural map $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$ corresponds to β_{ko} under the identification $\tilde{f}_4\mathsf{ko} \simeq \tilde{f}_4\mathsf{KO} \simeq \tilde{f}_4(\Sigma^{8,4}\mathsf{KO}) \simeq T^{\wedge 4} \wedge \tilde{f}_0\mathsf{KO} \simeq T^{\wedge 4} \wedge \mathsf{ko}$, where the first equivalence is induced by β_{KO} (periodicity of KO) and the second one is Proposition 1.9.

Step 2: *Setting up a spectral sequence that computes $|\pi_4(r_{\mathbb{R}}C)|$.* Under sufficiently nice assumptions, any (co)filtered object in a stable category has an associated strongly convergent spectral sequence computing the values of a fixed homological functor on this object. We recall the general theory in Subsection 4.2.4. We will consider the very effective filtration on $C = \mathsf{ko}/\tilde{f}_4\mathsf{ko}$, and after applying real realization, this will give us a filtration on $r_{\mathbb{R}}C$. The E^1-page of such a spectral sequence involves the homotopy groups of the successive cofibers of the maps in the filtered object. In our case, these will be the homotopy groups of the real realizations of the very effective slices of C, which are the first four very effective slices of ko. As we saw in Theorem 4.2, these slices have already been computed in [Bac17, Thm. 16].

Step 3: *Making explicit computations with this spectral sequence.* We computed the realizations of all the motivic spectra appearing as slices of ko in Section 3, except for $\tilde{s}_0\mathsf{ko}$. The only grasp we have on the latter is given by its decomposition as a cofiber sequence in Theorem 4.2. To compute the homotopy groups of $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$, we will use another spectral sequence, this time with respect to the filtration given by the effective homotopy t-structure. This is the content of Lemma 4.13. This allows us to prove $|\pi_4(r_{\mathbb{R}}C)| = 16$ in Lemma 4.16, using the first spectral sequence we mentioned. □

4.2.4. *Digression: spectral sequences with a multiplicative structure.*

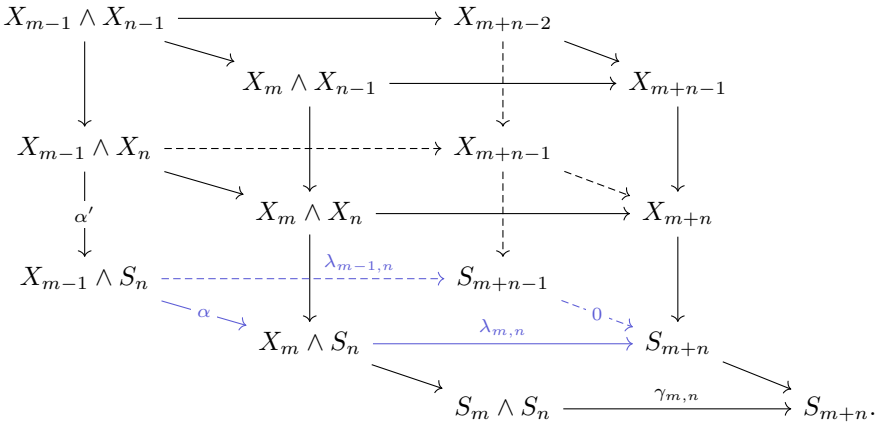
The proofs of Lemmas 4.13 and 4.16 needed for Proposition 4.9 will require us to use spectral sequences arguments. We therefore recall the necessary machinery.

Theorem 4.10. *Let $X := \cdots \to X_{-1} \to X_0 \to X_1 \to \cdots$ be a tower in Sp. Then, there is a spectral sequence $E^1_{p,q} = \pi_p(\mathsf{cof}(X_{-q-1} \to X_{-q}))$ converging conditionally to $\pi_p(\mathsf{cof}(\lim_n X_n \to \mathsf{colim}_n X_n))$. Moreover, if the spectral sequence is concentrated in the upper half-plane and each term is the source of only finitely many non-trivial differentials, then it converges strongly. Namely, for all $p, q \in \mathbb{Z}$, we have $E^\infty_{p,q} = \mathsf{coker}(\pi_p(X_{-q}) \to \pi_p(X_{-q+1}))$ and $\pi_p(\mathsf{colim}_n X_n) = \mathsf{colim}_q \mathsf{coker}(\pi_p(X_{-q}) \to \pi_p(\mathsf{colim}_n X_n))$.*

We call the cofibers $\mathsf{cof}(X_{-q-1} \to X_{-q})$ *the* subquotients *associated with the cofiltered object* $(X_n)_{n\in\mathbb{Z}}$.

Proof. Such a spectral sequence for cofiltered objects in a general stable category $\mathcal{C}$ is defined in [Lur17, Construction 1.2.2.6]. The conditional convergence is proven in [BH20, Lem. 6.16]. Then, the statement about strong convergence is [Boa99, Remark below Thm. 7.1]. □

If the tower of spectra we started with admits some kind of multiplicative structure, the spectral sequence also inherits a multiplicative structure, which can help to determine certain differentials in such spectral sequences. More precisely, assume that $(X_n)_{n\in\mathbb{Z}}$ admits a pairing taking the form of compatible maps $X_n \wedge X_m \to X_{n+m}$ for all $n, m \in \mathbb{Z}$. This descends to a pairing on the subquotients, given by the maps $\gamma_{n,m}$ induced between the cofibers of α and the zero map in the bottom face of the following diagram (where $S_n := \mathsf{cof}(X_{n-1} \to X_n)$ for all $n \in \mathbb{Z}$):



(5)

All vertical composites are cofiber sequences; the maps in blue in the bottom face are the maps induced on the cofibers by the top two layers of horizontal maps.

Proposition 4.11 ([Dug03, §6, Thm. 6.2])**.** *In the situation described above, the pairing on the subquotients itself descends to a pairing* $E^r_{p,q} \otimes E^r_{s,t} \to E^r_{p+s,q+t}$ *(for all* $r \geq 1$ *and* $p, q, s, t \in \mathbb{Z}$*) on the spectral sequence; and the differentials* d^r *satisfy the (graded) Leibniz rule with respect to this pairing.*

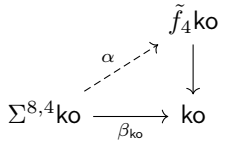
4.2.5. *Results used in the proof of Proposition 4.9.*

Lemma 4.12. *Let* $a \in \mathbb{Z}$ *with* $r_{\mathbb{R}}(\beta_{\mathsf{ko}}) = ax$ *(where* x *generates* $\pi_4(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}$*). Then* $a = \pm|\pi_4(r_{\mathbb{R}}C)|$*, where* C *is the cofiber of the natural map* $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$*.*

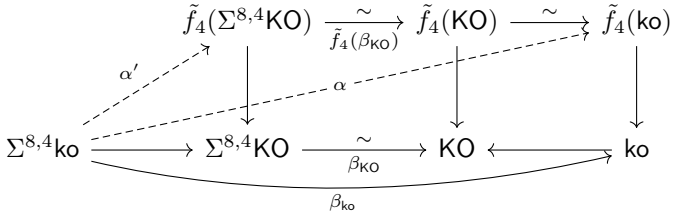
Proof. The cofiber sequence $\tilde{f}_4\mathsf{ko} \to \mathsf{ko} \to C$ induces a long exact sequence

$$\cdots \longrightarrow \pi_4(r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}) \longrightarrow \pi_4(r_{\mathbb{R}}\mathsf{ko}) \longrightarrow \pi_4(r_{\mathbb{R}}C) \longrightarrow \pi_3(r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}) \longrightarrow \cdots.$$

Since by definition $\Sigma^{8,4}\mathsf{ko} \in \mathsf{SH}(k)^{\mathsf{veff}}(4)$, there is a lift



where $\tilde{f}_4\mathsf{ko} \to \mathsf{ko}$ is the natural map. We claim that α is an equivalence. In particular, by Proposition 4.1, applying real realization and taking the fourth homotopy groups, the map $\pi_4(r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}) \to \pi_4(r_{\mathbb{R}}\mathsf{ko})$ is a morphism $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by a. Since $r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}$ is 4-connective by Lemma 1.11, in the long exact sequence above we have $\pi_3(r_{\mathbb{R}}\tilde{f}_4\mathsf{ko}) \cong 0$ and thus $\pi_4(r_{\mathbb{R}}C) \cong \mathbb{Z}/a$. Therefore, to finish the proof we only have to show that α is an equivalence. In the following diagram



the map α factors through the lift α' also obtained by very 4-effectiveness of $\Sigma^{8,4}\mathsf{ko}$. But α' is exactly the equivalence $T^{\wedge 4}\wedge \tilde{f}_0\mathsf{KO} \simeq \tilde{f}_4(T^{\wedge 4}\wedge \mathsf{KO})$ of Proposition 1.9. From the commutativity of the diagram, we deduce that α' is a composition of three equivalences, and thus an equivalence, as needed. □

Lemma 4.13. *The homotopy groups of $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$ are given for $k \geq 0$ by*

$$\pi_k(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } k \equiv 2 \bmod 4 \text{ or } k \equiv 3 \bmod 4 \\ 0 & \text{if } k \equiv 1 \bmod 4 \end{cases}$$

and in positive degrees divisible by 4, $\pi_k(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko})$ is an extension of two copies of $\mathbb{Z}/2$.

Remark 4.14. A posteriori, using the proof of Lemma 4.16, we get that $\pi_4(r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko}))$ is at the same time a quotient of a subgroup of $\pi_4(r_{\mathbb{R}}(C)) \cong \mathbb{Z}/16$ and an extension of two copies of $\mathbb{Z}/2$. Therefore, we have $\pi_4(r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})) \simeq \mathbb{Z}/4$.

Proof. We apply Theorem 4.10 with $X_n = r_{\mathbb{R}}((\tilde{s}_0\mathsf{ko})_{\geq_e -n})$ for all $n \in \mathbb{Z}$. We will see in Step 1 below that the spectral sequence obtained satisfies the additional assumptions for the strong convergence. The colimit $\mathsf{colim}_n X_n$ is eventually constant with value $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$, whereas the limit $\lim_n X_n$ is trivial because X_n is $-n$ connective for all $n \leq 0$. Indeed,

$$\pi_r(X_n) = \pi_r(r_{\mathbb{R}}((\tilde{s}_0\mathsf{ko})_{\geq_e -n})) = \pi_r(\Sigma^{-n} r_{\mathbb{R}}((\Sigma^n \tilde{s}_0\mathsf{ko})_{\geq_e 0})) = 0$$

for all $r < -n$ (because $(\Sigma^r \tilde{s}_0\mathsf{ko})_{\geq_e 0} \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} = \mathsf{SH}(k)^{\mathsf{veff}}$ by Proposition 1.5, and very effective spectra real realize to connective ones by Lemma 1.11). Since the limit of a tower of k-connective spectra is $(k-1)$-connective [GJ99, Prop. VI.2.15], $\lim_n X_n$ is ∞-connective and thus trivial. Thus, the spectral sequence converges strongly to $\pi_p(r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko}))$.

Step 1: *In the E^1-page, only the zeroth and first line have non-trivial terms, i.e., only the zeroth and first subquotients are non-trivial.* Indeed, we have by definition $\tilde{s}_0\mathsf{ko} \in \mathsf{SH}(\mathbb{R})^{\mathsf{veff}} = \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0}$ (by Proposition 1.5(i)), so that $(\tilde{s}_0\mathsf{ko})_{\geq_e 0} = \tilde{s}_0\mathsf{ko}$. By Claim (1) in the proof of [Bac17, Lem. 11], we have $(\tilde{s}_0\mathsf{ko})_{\geq_e 1} \simeq s_0(\mathsf{ko}_{\geq 1})$, and the latter is the first term in the cofiber sequence of Theorem 4.2, namely $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$. The natural map $(\tilde{s}_0\mathsf{ko})_{\geq_e 1} \to \tilde{s}_0\mathsf{ko}$ is exactly the map $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \to \tilde{s}_0$ appearing in this cofiber sequence. In particular, the 0-th subquotient is nothing but the third term of this cofiber sequence, $\widetilde{\mathsf{H}\mathbb{Z}}$. Let us denote $s'_1\mathsf{ko} := (\tilde{s}_0\mathsf{ko})_{\geq_e 1} \simeq \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$ and $s'_0\mathsf{ko} := (\tilde{s}_0\mathsf{ko})/((\tilde{s}_0\mathsf{ko})_{\geq_e 1}) \simeq \widetilde{\mathsf{H}\mathbb{Z}}$. The first subquotient is actually $s'_1\mathsf{ko}$ since $(\tilde{s}_0\mathsf{ko})_{\geq_e k} \simeq 0$ for all $k \geq 2$. Indeed, in the cofiber sequence

$$(\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2)_{\geq_e k} \to (\tilde{s}_0\mathsf{ko})_{\geq_e k} \to (\widetilde{\mathsf{H}\mathbb{Z}})_{\geq_e k},$$

the two outer terms are zero, because by Proposition 3.3 we have $\mathsf{H}\mathbb{Z}/2, \widetilde{\mathsf{H}\mathbb{Z}} \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$, in particular $(\widetilde{\mathsf{H}\mathbb{Z}})_{\geq_e k} \simeq 0$ for all $k \geq 1$ and $(\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2)_{\geq_e k} \simeq \Sigma^{1,0}(\mathsf{H}\mathbb{Z}/2_{\geq_e k-1}) \simeq 0$ for all $k \geq 2$.

Step 2: *We write down the E^1-page.* By Propositions 3.6 and 3.11, we know the homotopy groups of $r_{\mathbb{R}}(s'_0\mathsf{ko})$ and $r_{\mathbb{R}}(s'_1\mathsf{ko})$. Therefore, the E^1-page of our spectral sequence for $\pi_*(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko})$ reads as follows:

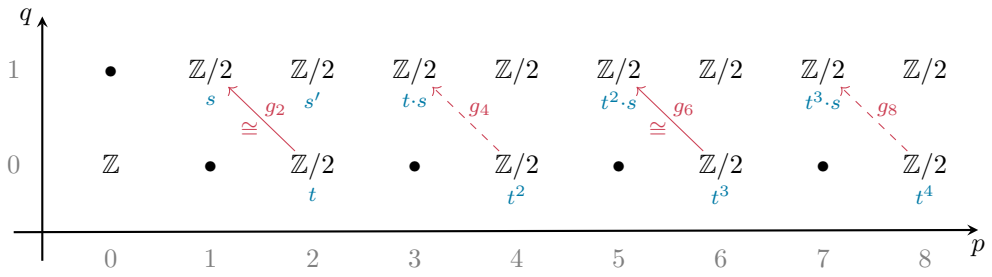


FIGURE 1. E^1-page for the spectral sequence for $r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}$ with respect to the effective homotopy t-structure.

The bullets "•" represent trivial groups. In red are some d^1 differentials, we will explain how to determine them just below. Dotted arrows mean that the differential is trivial. The elements written in blue under some groups in the E^1-page are generators of these groups; we will see in Step 4 why it is

indeed the case.

Step 3: *We identify the differential g_2.* To begin with, note that we must have $\pi_1(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}) \cong 0$. Indeed, this group is the $E^1_{1,0}$-term in the spectral sequence given by the theorem for $X'_n = r_{\mathbb{R}}(\tilde{f}_{-n}\mathsf{ko})$ (by convention $\tilde{f}_n\mathsf{ko} = \mathsf{ko}$ for $n < 0$) (the assumptions are the theorem are verified in this case by the same arguments as in the beginning of this proof). We obtain a spectral sequence concentrated in the first quadrant with $E^1_{p,q} = \pi_p(\tilde{s}_q\mathsf{ko})$ for all $p, q \in \mathbb{N}$, converging strongly to $\pi_p(r_{\mathbb{R}}\mathsf{ko})$. In particular, the $E^1_{1,0}$-term will never receive a non-trivial differential. Moreover, since the first column of this same spectral contains only zeroes except in position $(0,0)$ (we have $\pi_0(r_{\mathbb{R}}\tilde{s}_i\mathsf{ko}) \cong 0$ for all $i \geq 1$), it will never be the source of a non-trivial differential either, and will survive to the E^∞-page. But since $\pi_1(r_{\mathbb{R}}\mathsf{ko}) \cong 0$ by Proposition 4.1, nothing in the first column can survive. This means that $\pi_1(r_{\mathbb{R}}\tilde{s}_0\mathsf{ko}) \cong 0$. In particular, in the spectral sequence for $r_{\mathbb{R}}(\tilde{s}_0\mathsf{ko})$ and the effective homotopy t-structure, $E^1_{1,1} \cong \mathbb{Z}/2$ must eventually die. Since it cannot be the source of any non-trivial differential, the only possibility is that it receives a non-trivial differential g_2 from $E^1_{2,0} \cong \mathbb{Z}/2$, which must then be an isomorphism.

Step 4: *We use the multiplicative structure on this spectral sequence to compute the next differentials.* In the situation of Proposition 4.11, let $E = \tilde{s}_0\mathsf{ko}$ and $X_n := r_{\mathbb{R}}(E_{\geq_e -n})$ for all $n \in \mathbb{Z}$. The pairings $X_{-n} \wedge X_{-m} \to X_{-n-m}$ are given by the real realizations of the compositions $E_{\geq_e n} \wedge E_{\geq_e m} \to E \wedge E \to E$ of the natural maps with the multiplication on E (the latter is itself induced by the pairing in the tower for the very effective slice filtration on ko, as we will see below), which lifts to $E_{\geq_e n+m}$. Indeed, we have

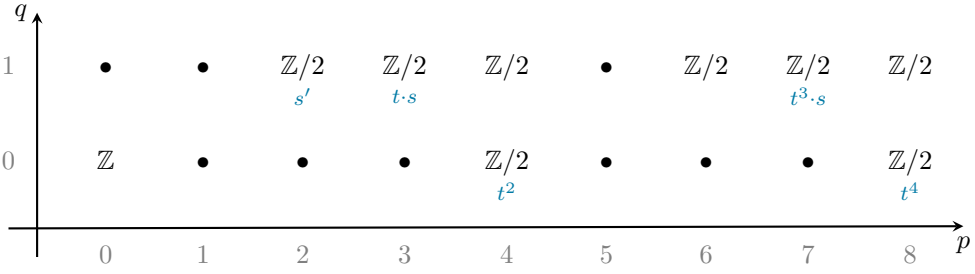
$$\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq n} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq m} = \Sigma^{n,0}\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} \wedge \Sigma^{m,0}\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} \subseteq \Sigma^{m+n,0}\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0} = \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq m+n}$$

because the effective homotopy t-structure is compatible with the symmetric monoidal structure (in the sense of [AN21, Def. A.10]). In particular, we have pairings

$$\widetilde{\mathsf{H}\mathbb{Z}} \wedge \widetilde{\mathsf{H}\mathbb{Z}} \simeq s'_0\mathsf{ko} \wedge s'_0\mathsf{ko} \to s'_0\mathsf{ko} = \widetilde{\mathsf{H}\mathbb{Z}},$$
$$\widetilde{\mathsf{H}\mathbb{Z}} \wedge \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \simeq s'_0\mathsf{ko} \wedge s'_1\mathsf{ko} \to s'_1\mathsf{ko} = \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2.$$

We defer to Lemma 4.15 below the proof that these are exactly the pairings induced, respectively, by the ring structure on $\widetilde{\mathsf{H}\mathbb{Z}}$ and the composition $\widetilde{\mathsf{H}\mathbb{Z}} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z}/2$ induced by the natural map from Milnor–Witt K-theory to Milnor K-theory and the $\mathsf{H}\mathbb{Z}$-algebra structure on $\mathsf{H}\mathbb{Z}/2$. Then, the pairing $r_{\mathbb{R}}(s'_0\mathsf{ko}) \wedge r_{\mathbb{R}}(s'_0\mathsf{ko}) \to r_{\mathbb{R}}(s'_0\mathsf{ko})$ on the subquotients is given in homotopy by the ring structure of $\pi_*(r_{\mathbb{R}}\widetilde{\mathsf{H}\mathbb{Z}}) = \mathbb{Z}[t^2]/(2t^2)$ (Proposition 3.11), with $\widetilde{\mathsf{H}\mathbb{Z}} \to \mathsf{H}\mathbb{Z}/2$ killing 2. The pairing $r_{\mathbb{R}}(s'_0\mathsf{ko}) \wedge r_{\mathbb{R}}(s'_1\mathsf{ko}) \to r_{\mathbb{R}}(s'_1\mathsf{ko})$ is given in homotopy, after desuspension, by $\mathbb{Z}[t^2]/(2t^2) \otimes \mathbb{Z}/2[t] \to \mathbb{Z}/2[t^2] \otimes \mathbb{Z}/2[t] \to \mathbb{Z}/2[t]$, where the first map is the quotient and the second one is multiplication (viewing $\mathbb{Z}/2[t^2] \subseteq \mathbb{Z}/2[t]$ by Proposition 3.6). This explains why the generators are the ones displayed in the spectral sequence in Figure 1. Since g_2 is an isomorphism, we have $d^1(t) = s$. It follows that g_{2k} is zero for k even and an isomorphism for k odd. Indeed, by the Leibniz rule $g_{2k}(t^k) = d^1(t^k) = kt^{k-1}d^1(t) \in \mathbb{Z}/2$ is 0 if k is even and equals $t^{k-1} \cdot s$ otherwise (i.e., a generator in the target).

Step 5: *We conclude.* The previous steps show that the E^1-page of our spectral sequence has the form displayed in Figure 1. Therefore, the E^2-page reads as follows:



and for degree reasons, no non-trivial differential can exist (on the E^2-page, differentials have degree (-1,2)). Thus, the spectral sequence collapses, and this finishes the proof. □

Lemma 4.15. *For any $E \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$, there is an equivalence*

$$\mathsf{Map}_{\mathsf{SH}(\mathbb{R})}(\widetilde{\mathsf{H}\mathbb{Z}} \wedge E, E) \xrightarrow{\simeq} \mathsf{Map}_{\mathsf{SH}(\mathbb{R})}(E, E)$$

induced by precomposition by the map $E \simeq \mathbb{1} \wedge E \to \widetilde{\mathsf{H}\mathbb{Z}} \wedge E$ *(given by the unit of the* $\mathcal{E}_\infty$*-ring* $\widetilde{\mathsf{H}\mathbb{Z}}$*).*

In particular, the pairings $\widetilde{\mathsf{H}\mathbb{Z}} \wedge \widetilde{\mathsf{H}\mathbb{Z}} \to \widetilde{\mathsf{H}\mathbb{Z}}$ *and* $\widetilde{\mathsf{H}\mathbb{Z}} \wedge \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z}/2$ *for the multiplicative spectral sequence of Figure 1 (before taking real realization and homotopy group) are given, respectively, by the multiplication of* $\widetilde{\mathsf{H}\mathbb{Z}}$ *and the composition* $\widetilde{\mathsf{H}\mathbb{Z}} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z}/2$ *induced by the natural map from Milnor–Witt K-theory to Milnor K-theory and the* $\mathsf{H}\mathbb{Z}$*-algebra structure on* $\mathsf{H}\mathbb{Z}/2$*.*

Proof. First, we claim that, for any $F, G \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0}$, we have $(F \wedge G)_{\leq_e 0} \simeq (F_{\leq_e 0} \wedge G_{\leq_e 0})_{\leq_e 0}$ via the natural maps $F \to F_{\leq_e 0}$ and $G \to G_{\leq_e 0}$. Indeed, almost by definition of a t-structure, we have in $\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 0}$ a cofiber sequence $F_{\geq_e 1} \wedge G \to F \wedge G \to F_{\leq_e 0} \wedge G$. The associated long exact sequence then gives an equivalence $(F \wedge G)_{\leq_e 0} \xrightarrow{\simeq} (F_{\leq_e 0} \wedge G)_{\leq_e 0}$ (since $F_{\geq_e 1} \wedge G \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq_e 1} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq_e 0} \subseteq \mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq_e 1}$). Applying the same argument while exchanging the roles of F and G, we obtain our claim.

Let $E \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$. Using the previous claim and $\widetilde{\mathsf{H}\mathbb{Z}} = \mathbb{1}_{\leq_e 0}$ [Bac17, Lem. 12], we have

$$\begin{aligned}\mathsf{Map}_{\mathsf{SH}(\mathbb{R})}(\widetilde{\mathsf{H}\mathbb{Z}} \wedge E, E) &\simeq \mathsf{Map}_{\mathsf{SH}(\mathbb{R})}((\widetilde{\mathsf{H}\mathbb{Z}} \wedge E)_{\leq_e 0}, E) \simeq \mathsf{Map}_{\mathsf{SH}(\mathbb{R})}((\mathbb{1}_{\leq_e 0} \wedge E_{\leq_e 0})_{\leq_e 0}, E) \\ &\simeq \mathsf{Map}_{\mathsf{SH}(\mathbb{R})}((\mathbb{1} \wedge E)_{\leq_e 0}, E) \simeq \mathsf{Map}_{\mathsf{SH}(\mathbb{R})}(\mathbb{1} \wedge E, E).\end{aligned}$$

This equivalence of mapping spaces is induced by precomposition with $\mathbb{1} \to \mathbb{1}_{\leq_e 0} \simeq \widetilde{\mathsf{H}\mathbb{Z}}$ (tensored with E). Indeed, there is a commutative diagram

$$\begin{array}{ccccc} \mathbb{1} \wedge E & \xrightarrow{\tau_1} & \mathbb{1}_{\leq_e 0} \wedge E_{\leq_e 0} \simeq \widetilde{\mathsf{H}\mathbb{Z}} \wedge E & \xrightarrow{\vartheta} & E_{\leq_e 0} \simeq E \\ {\scriptstyle \tau_2}\downarrow & & \downarrow & \nearrow_{\vartheta'} & \\ (\mathbb{1} \wedge E)_{\leq_e 0} & \xrightarrow{\tau_3} & (\mathbb{1}_{\leq_e 0} \wedge E_{\leq_e 0})_{\leq_e 0}. & & \end{array}$$

where all maps in the square are induced by the natural maps in the truncation. The equivalence above sends a map ϑ to the composite $\vartheta' \circ \tau_3 \circ \tau_2 \simeq \vartheta \circ \tau_1$.

We now apply this result to $E = \widetilde{\mathsf{H}\mathbb{Z}}$, or $E = \mathsf{H}\mathbb{Z}/2$ (recall that $\widetilde{\mathsf{H}\mathbb{Z}}, \mathsf{H}\mathbb{Z}/2 \in \mathsf{SH}(\mathbb{R})^{\mathsf{eff},\heartsuit}$ by Proposition 3.3). Since $\mathsf{SH}(\mathbb{R})(\widetilde{\mathsf{H}\mathbb{Z}} \wedge \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2, \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2) \simeq \mathsf{SH}(\mathbb{R})(\widetilde{\mathsf{H}\mathbb{Z}} \wedge \mathsf{H}\mathbb{Z}/2, \mathsf{H}\mathbb{Z}/2)$, we only have to show that the pairings induced on the subquotients of the tower correspond to the unitor after precomposition by the natural map $\mathbb{1} \to \mathbb{1}_{\leq_e 0} \simeq \widetilde{\mathsf{H}\mathbb{Z}}$. This can be proven by chasing through the construction of the pairing induced on the subquotients; which is ultimately obtained from the multiplication on ko (and the latter is unital). More precisely, the unit $\mathbb{1} \to \mathbb{1}_{\leq_e 0} = \widetilde{\mathsf{H}\mathbb{Z}}$ is exactly the one induced by

$$\mathbb{1} \longrightarrow \mathsf{ko} \longrightarrow \tilde{s}_0\mathsf{ko} = f_0(\tilde{s}_0\mathsf{ko}) \longrightarrow f_0((\tilde{s}_0\mathsf{ko})_{\leq 0}) = f_0(\underline{\pi}_0(\tilde{s}_0\mathsf{ko})) = s'_0\mathsf{ko}.$$

To check that $\mathbb{1} \wedge s'_0\mathsf{ko} \to s'_0\mathsf{ko} \wedge s'_0\mathsf{ko} \to s'_0\mathsf{ko}$ is the unitor, for example, using Diagram (5), we want to show that it holds true for $\mathbb{1} \wedge (\tilde{s}_0\mathsf{ko})_{\geq_e i} \to (\tilde{s}_0\mathsf{ko})_{\geq_e i}$ for $i = 0, 1$. Since these pairings are induced by one we had on $\tilde{s}_0$, which is induced from that on the very effective filtration, we want to show it for the maps $\mathbb{1} \wedge \tilde{f}_i\mathsf{ko} \to \tilde{f}_i\mathsf{ko}$ for $i = 0, 1$, and these where ultimately induced by $\mathbb{1} \wedge \mathsf{ko} \to \mathsf{ko} \wedge \mathsf{ko} \to \mathsf{ko}$, whence the claim follows. □

Lemma 4.16. *With the same notation as in Lemma 4.12, we have* $|\pi_4(r_\mathbb{R}C)| = 16$*.*

Proof. Applying Theorem 4.10 again, this time with $X_n = r_\mathbb{R}(\tilde{f}_{-n}\mathsf{ko})$ for all $n \in \mathbb{Z}$ (by convention $\tilde{f}_n\mathsf{ko} = \mathsf{ko}$ for $n < 0$), we obtain a spectral sequence concentrated in the first quadrant, with $E^1_{p,q} = \pi_p(\tilde{s}_q\mathsf{ko})$ for all $p, q \in \mathbb{N}$. As we will see in Figure 2, the assumptions for strong convergence are satisfied. We have $\mathsf{colim}_n X_n \simeq r_\mathbb{R}\mathsf{ko}$, and $\lim_n X_n \simeq 0$ by the same argument as in the beginning of the proof of Lemma 4.13, since a very n-effective spectra is in particular n-connective in the effective homotopy t-structure. Indeed, we have $\mathsf{SH}(k)^{\mathsf{veff}}(n) := T^{\wedge n} \wedge \mathsf{SH}(k)^{\mathsf{veff}} \subseteq \Sigma^n\mathsf{SH}(k)^{\mathsf{veff}} = \mathsf{SH}(k)^{\mathsf{eff}}_{\geq n}$ by Proposition 1.5(i). Therefore, our spectral sequence converges strongly to $\pi_p(r_\mathbb{R}\mathsf{ko})$. We know the latter groups by Proposition 4.1, and we will use this to compute several differentials in the E^1-page. Then, since the same spectral sequence for ko replaced with C is given by the truncation above the third line, this will give us information about the homotopy groups of $r_\mathbb{R}(C)$.

Using the explicit description of the very effective slices of ko (Theorem 4.2) and the computation of the real realizations of these slices (Propositions 3.6 and 3.11, Lemma 4.13), the E^1-page of our spectral sequence reads as in Figure 2 below. The notation is as in Figure 1. We will explain just below how to determine the differentials, but let us first see how this implies the result. The spectral sequence for $r_\mathbb{R}C = r_\mathbb{R}(\mathsf{ko}/\tilde{f}_4\mathsf{ko})$ is the truncation of that for $r_\mathbb{R}\mathsf{ko}$ above the third horizontal line (indeed, $\tilde{s}_iC = \tilde{s}_i\mathsf{ko}$ for $0 \leq i \leq 3$, and 0 otherwise). In particular, the differentials must be the same and in the fourth column, all terms on the E^1-page survive to the $\mathcal{E}_\infty$-page. Therefore, $\pi_4(r_\mathbb{R}C)$ is an extension of four

copies of $\mathbb{Z}/2$; it has cardinality 16 (we can even identify it as $\mathbb{Z}/16$ because we saw in the proof of Lemma 4.12 that it was cyclic), as desired.

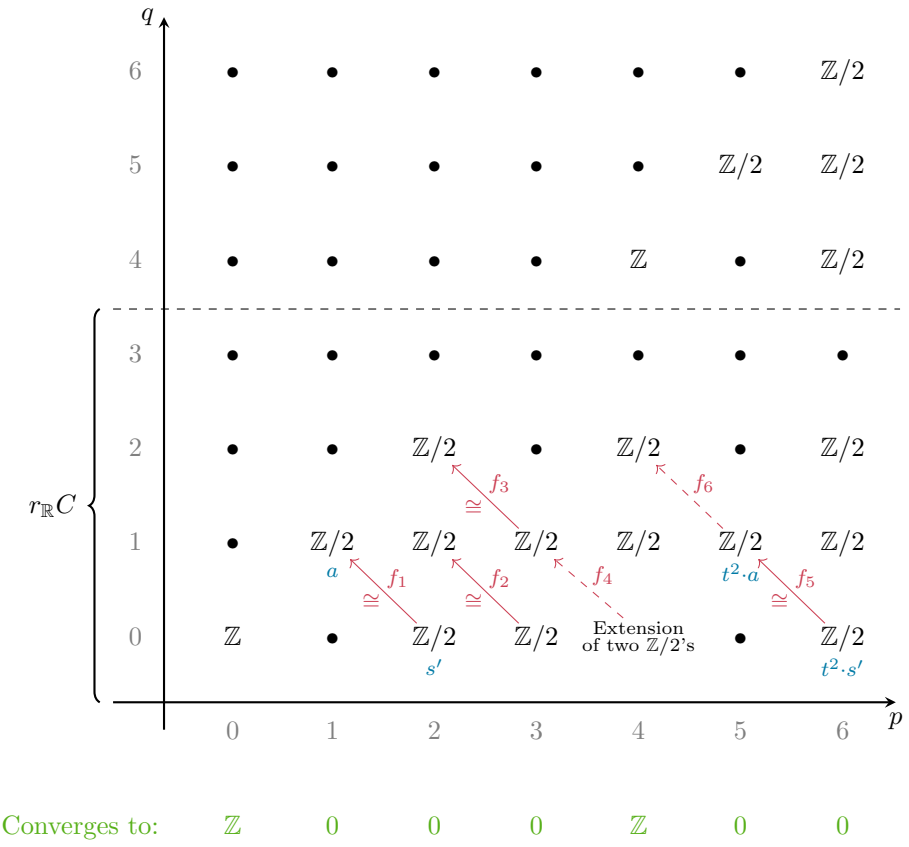


FIGURE 2. E^1-page of the spectral sequence for $r_{\mathbb{R}}\mathsf{ko}$ with respect to the very effective slice filtration.

To conclude the proof, we only have to explain why the differentials are the ones displayed in Figure 2. First, note that by Proposition 4.1, the spectral sequence for $r_{\mathbb{R}}\mathsf{ko}$ converges to the groups displayed in green above, i.e., to $\mathbb{Z}$ in every degree divisible by 4, and 0 otherwise.

- No non-zero differential with source $E^1_{1,1} = \mathbb{Z}/2$ can exist. So it must receive a non-trivial differential on some page; the only possibility is that $d^1 : E^1_{2,0} \to E^1_{1,1}$, namely f_1, is an isomorphism.
- The same argument shows that f_2 is an isomorphism.
- The $E^1_{2,2}$-term could a priori also receive a non-trivial differential d^2 from $E^2_{3,0}$ on the E^2-page; but since f_2 is an isomorphism, $E^2_{3,0} = 0$. Thus, f_3 must be an isomorphism.
- Since $d^1 \circ d^1 = 0$, we have $f_3 \circ f_4 = 0$, so by the previous bullet point $f_4 = 0$. The $E^1_{4,0}$-term then survives to the E_∞-page. So does the $E^1_{4,1}$-term (it cannot be reached by any non-trivial differential).

The multiplicative structure on the spectral sequence provides the missing information to determine f_5 and f_6. The multiplicative structure comes this time from the pairing $\tilde{f}_n\mathsf{KO} \wedge \tilde{f}_m\mathsf{KO} \to \tilde{f}_{n+m}\mathsf{KO}$ obtained as follows: for any $n, m \geq 0$, the composition $\tilde{f}_n\mathsf{KO} \wedge \tilde{f}_m\mathsf{KO} \to \mathsf{KO} \wedge \mathsf{KO} \to \mathsf{KO}$ given by the natural maps, followed by multiplication, lifts to $\tilde{f}_{n+m}\mathsf{ko}$, because

$$\begin{aligned}\mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(n) \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(m) &= (T^{\wedge n} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}) \wedge (T^{\wedge m} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}) \\ &\subseteq (T^{\wedge(n+m)} \wedge \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}) = \mathsf{SH}(\mathbb{R})^{\mathsf{veff}}(n+m).\end{aligned}$$

The compatibility between these pairings come from the $\mathcal{E}_\infty$-ring axioms for KO. Then the pairing descends to the slices as Diagram (5). We claim that the pairing $\tilde{s}_0\mathsf{ko} \wedge \tilde{s}_1\mathsf{ko} \to \tilde{s}_1\mathsf{ko}$ factors through $(\tilde{s}_0\mathsf{ko})/(s'_1\mathsf{ko}) \wedge \tilde{s}_1\mathsf{ko} \simeq s'_0\mathsf{ko} \wedge \tilde{s}_1\mathsf{ko}$. Indeed, the latter is the cofiber of $s'_1\mathsf{ko} \wedge \tilde{s}_1\mathsf{ko} \to \tilde{s}_0\mathsf{ko} \wedge \tilde{s}_1\mathsf{ko}$, whose post composition with the pairing is zero. This holds because $\Sigma_T^{-1}(s'_1\mathsf{ko} \wedge \tilde{s}_1\mathsf{ko} \to \tilde{s}_1\mathsf{ko})$ is a map $\Sigma^{1,0}\mathsf{H}\mathbb{Z}/2 \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z}/2$, so it must be zero by the axioms of a t-structure (the source belongs to $\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\geq 1}$ whereas the right-hand side belongs to $\mathsf{SH}(\mathbb{R})^{\mathsf{eff}}_{\leq 0}$ (Proposition 3.3)). Moreover, the map

$\tilde{s}_0\mathsf{ko} \wedge s'_1\mathsf{ko} \to \tilde{s}_0\mathsf{ko} \wedge \tilde{s}_0\mathsf{ko} \to \tilde{s}_0\mathsf{ko}$ induced by the natural map and the pairing lifts to $s'_1\mathsf{ko}$ on the target, and then by the same argument as above this lift factors through $s'_0\mathsf{ko} \wedge s'_1\mathsf{ko}$. Then, since $\tilde{s}_1\mathsf{ko} \simeq \Sigma^{2,1}\mathsf{H}\mathbb{Z}/2$, we can use the exact same strategy as we did for $s'_1\mathsf{ko} \simeq \Sigma^{1,0}\mathsf{H}\mathbb{Z}/2$ to show that $s'_0\mathsf{ko} \wedge \tilde{s}_1\mathsf{ko} \to \tilde{s}_1\mathsf{ko}$ is the $(2,1)$ suspension of the pairing $\widetilde{\mathsf{H}\mathbb{Z}} \wedge \mathsf{H}\mathbb{Z}/2 \to \mathsf{H}\mathbb{Z}/2$ previously considered. In particular, we deduce the following:

- Let us also denote by s' a generator of $\pi_2(r_\mathbb{R}\tilde{s}_0\mathsf{ko}) \cong \mathbb{Z}/2$ (it comes from the generator s' of $\pi_2(r_\mathbb{R}s'_1\mathsf{ko})$ in Figure 1). Then, a preimage in $\pi_4(r_\mathbb{R}\tilde{s}_0\mathsf{ko})$ for $t^2 \in \pi_4(r_\mathbb{R}s'_0\mathsf{ko})$ acts upon s' by sending it to the image of $t^2 \cdot s' \in \pi_6(r_\mathbb{R}s'_1\mathsf{ko})$ in $\pi_6(r_\mathbb{R}\tilde{s}_0\mathsf{ko}) \cong \mathbb{Z}/2$, which is a generator, as we can see from the spectral sequence for $r_\mathbb{R}\tilde{s}_0\mathsf{ko}$ (Figure 1).
- Let a be a generator of $\pi_1(r_\mathbb{R}\tilde{s}_1\mathsf{ko}) \cong \mathbb{Z}/2$. Then a preimage in $\pi_4(r_\mathbb{R}\tilde{s}_0\mathsf{ko})$ for $t^2 \in \pi_4(r_\mathbb{R}s'_0\mathsf{ko})$ acts upon a by sending it to a generator in $\pi_5(r_\mathbb{R}\tilde{s}_1\mathsf{ko}) \cong \mathbb{Z}/2$ (this is the action of $r_\mathbb{R}\widetilde{\mathsf{H}\mathbb{Z}}$ on $r_\mathbb{R}\Sigma^{2,1}\mathsf{H}\mathbb{Z}/2$ as we have seen it before, for the spectral sequence in Figure 1).
- Therefore, since by the Leibniz rule for d^1, we have
$$f_5(t^2 \cdot s') = d^1(t^2 \cdot s') = t^2 \cdot d^1(s') + d^1(t^2) \cdot s' = t^2 \cdot a \in E^1_{5,1} \cong \mathbb{Z}/2$$
(since f_1 is an isomorphism and f_4 is trivial), f_5 maps a generator to a generator and is an isomorphism. In particular, since $d^1 \circ d^1 = 0$, we also get $f_6 = 0$, and the $E^1_{4,2}$-term survives to the E_∞-page.

This finishes the proof. □

4.3. **Comparison with the L-theory of $\mathbb{R}$.**

Theorem 4.17. *There is an equivalence of $\mathcal{E}_1$-rings*
$$r_\mathbb{R}\mathsf{ko} \simeq \mathsf{L}(\mathbb{R})_{\geq 0}$$
between the real realization of the very effective Hermitian K-theory spectrum and the connective cover of the L-theory spectrum of $\mathbb{R}$.

Proof. It suffices to prove that the 2-local fracture square for $\mathsf{L}(\mathbb{R})_{\geq 0}$ is the same as in Theorem 4.8. Such a fracture square appears in [HLN21, p3] (with a typo[1]). We compute the fracture square for $L(\mathbb{R})_{\geq 0}$ following the same strategy as for $r_\mathbb{R}(\mathsf{ko})$ (Theorem 4.8).

To identify the localization away from 2, we consider the same map $\mathsf{ko}^{\mathsf{top}} \to \mathsf{L}(\mathbb{R})_{\geq 0}$ as in the proof of Theorem 4.8, from [LNS23]; recall that this map is an equivalence after localizing at 2, and induces multiplication by 8 on the fourth homotopy groups before localization. We post-compose it with the automorphism of $\mathsf{ko}^{\mathsf{top}}[1/2]$ given by the (connective cover of the) second Adams operation ψ^2 as in the proof of Theorem 4.8, and obtain $\mathsf{L}(\mathbb{R})_{\geq 0} \to \mathsf{ko}^{\mathsf{top}}[1/2]$ sending b to $\beta_4/2$ in the fourth homotopy groups.

To identify the localization at (2), we consider the map of $\mathcal{E}_\infty$-rings $\mathsf{MSO} \to \mathsf{L}(\mathbb{R})$ from [HLN21, Thm. 3.4 and Rmk. 3.5]. By [HLN21, Cor. 3.7], $\mathsf{MSO}_{(2)}$ receives an $\mathcal{E}_2$-map from $\mathsf{H}\mathbb{Z}_{(2)}$. Therefore, $\mathsf{L}(\mathbb{R})_{(2)}$ receives an $\mathcal{E}_2$-map from $\mathsf{H}\mathbb{Z}_{(2)}$. By Proposition 1.19(*iii*), it then also receives a map of $\mathcal{E}_1$-rings from the free $\mathcal{E}_1$-$\mathsf{H}\mathbb{Z}_{(2)}$-algebra $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$, sending t^4 to b in π_4. Since the domain is connective, this map lifts to $\mathsf{L}(\mathbb{R})_{\geq 0}$. The ring structure on the homotopy groups on both sides implies that this map is an equivalence (indeed, by [HLN21, Prop. 4.1], $\pi_*(\mathsf{L}(\mathbb{R})) \cong \mathbb{Z}[b]$ with $|b| = 4$, and by Proposition 1.21(*ii*), $\pi_*(\mathsf{H}\mathbb{Z}_{(2)}[t^4]) \cong \mathbb{Z}[t^4]$ is the same ring).

For the rationalization, by Remark 4.7 we have $\mathsf{ko}^{\mathsf{top}}[1/2]_\mathbb{Q} \simeq \mathsf{H}\mathbb{Q}[t^4]$ via the Chern character map.

Finally, we have to determine what the map $\mathsf{H}\mathbb{Z}_{(2)}[t^4] \to \mathsf{H}\mathbb{Q}[t^4]$ in the pullback square is, corresponding to our choices of identifications. Studying the actions of these maps on the fourth homotopy groups, we see that the composite $\mathsf{L}(\mathbb{R})_{\geq 0} \to \mathsf{ko}^{\mathsf{top}}[1/2] \to \mathsf{H}\mathbb{Q}[t^4]$ maps b to t^4. Since the map $\mathsf{L}(\mathbb{R})_{\geq 0} \to \mathsf{H}\mathbb{Z}_{(2)}[t^4]$ we have chosen sends b to t^4, the bottom horizontal map must send t^4 to t^4. □

[1]Private communication with Markus Land.

Appendix A. Day convolution on ∞-categories of presheaves

The category of functors between two symmetric monoidal categories can be endowed with a symmetric monoidal structure, called Day convolution.

Theorem A.1 ([Gla16, Prop. 2.11] and [Lur17, Ex. 2.2.6.9])**.** *Let $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ be symmetric monoidal categories. Assume that $\mathcal{D}$ has all small colimits and that the tensor product on $\mathcal{D}$ preserves them in each variable separately. Then the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ admits a symmetric monoidal structure $\mathsf{Fun}(\mathcal{C},\mathcal{D})^\otimes$, called* Day convolution*. It has in particular the property that there is an equivalence of categories*

$$\mathsf{CAlg}(\mathsf{Fun}(\mathcal{C},\mathcal{D})^\otimes) \simeq \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes).$$

For any symmetric monoidal category $\mathcal{C}^\otimes$, in the particular case $\mathcal{D}^\otimes = \mathsf{Spc}^\times$, Day convolution endows the category of presheaves $\mathcal{P}(\mathcal{C})$ with a symmetric monoidal structure (because the opposite of $\mathcal{C}$ has an induced symmetric monoidal structure [Lur17, Rmk. 2.4.2.7]). This symmetric monoidal structure has the following universal property:

Proposition A.2 ([Lur17, Cor. 4.8.1.12 and Rmk. 4.8.1.13])**.** *Let $\mathcal{C}^\otimes$ be a symmetric monoidal category. Day convolution is the essentially unique symmetric monoidal structure on the category $\mathcal{P}(\mathcal{C})$ such that:*
(1) the tensor product preserves colimits in both variables separately,
(2) and the Yoneda embedding $y : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ can be extended into a symmetric monoidal functor $y^\otimes$.

Proposition A.3. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal category, and consider $\mathcal{P}(\mathcal{C})^\otimes$ with the Day convolution symmetric monoidal structure. Let $\mathcal{D}^\otimes \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$.*

Then, the symmetric monoidal Yoneda embedding $y^\otimes : \mathcal{C}^\otimes \to \mathcal{P}(\mathcal{C})^\otimes$ induces an equivalence

$$- \circ y^\otimes : \mathsf{Fun}^{\mathsf{L},\mathsf{lax}}(\mathcal{P}(\mathcal{C})^\otimes,\mathcal{D}^\otimes) \xrightarrow{\simeq} \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes),$$

where $\mathsf{Fun}^{\mathsf{L},\mathsf{lax}}$ denotes colimit-preserving lax symmetric monoidal functors, and $\mathsf{Fun}^{\mathsf{lax}}$ all lax symmetric monoidal functors.

Definition A.4. In the setting of Proposition A.3, let $\mathsf{LKE}^\otimes$ be an inverse for the equivalence $- \circ y^\otimes$. Given $F \in \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$, we say that the lax symmetric monoidal functor $\mathsf{LKE}^\otimes(F) : \mathcal{P}(\mathcal{C})^\otimes \to \mathcal{D}^\otimes$ is *left Kan extended as a lax symmetric monoidal functor* from F.

Proof of Proposition A.3. Day convolution makes $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})$ and $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ into symmetric monoidal categories $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})^\otimes$ and $\mathsf{Fun}(\mathcal{C},\mathcal{D})^\otimes$, respectively. By [Nik16, Cor. 3.8], precomposition by a lax symmetric monoidal functor induces a lax symmetric monoidal functor with respect to the Day convolution on the functor categories. In our case, this implies that precomposition with the symmetric monoidal Yoneda embedding induces a lax symmetric monoidal functor $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})^\otimes \to \mathsf{Fun}(\mathcal{C},\mathcal{D})^\otimes$. By the usual universal property of the presheaf category (free cocompletion), the underlying functor restricts to an equivalence $\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D}) \to \mathsf{Fun}(\mathcal{C},\mathcal{D})$. In particular, $\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D})$ is a symmetric monoidal subcategory $\mathsf{Fun}(\mathcal{P}(\mathcal{C}),\mathcal{D})$, equivalent as a symmetric monoidal category to $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.

Therefore, precomposition by the Yoneda embedding induces an equivalence

$$\mathsf{CAlg}(\mathsf{Fun}^{\mathsf{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D})^\otimes) \xrightarrow{\simeq} \mathsf{CAlg}(\mathsf{Fun}(\mathcal{C},\mathcal{D})^\otimes).$$

By Theorem A.1, the left and right-hand sides are respectively equivalent to $\mathsf{Fun}^{\mathsf{L},\mathsf{lax}}(\mathcal{P}(\mathcal{C})^\otimes,\mathcal{D}^\otimes)$ and $\mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$. Indeed, for the first one, note that if $\mathcal{E}^\otimes$ is a symmetric monoidal category with a symmetric monoidal full subcategory $\mathcal{F}^\otimes$, then the categories of commutative algebras in $\mathcal{F}$ is that of commutative algebras in $\mathcal{E}$ whose underlying object lies in $\mathcal{F}$. This follows from the description of commutative algebras in terms of the representing coCartesian fibrations. □

Proposition A.5. *The equivalence of Proposition A.3 restricts to an equivalence:*

$$- \circ y^\otimes : \mathsf{Fun}^{\mathsf{L},\otimes}(\mathcal{P}(\mathcal{C})^\otimes,\mathcal{D}^\otimes) \xrightarrow{\simeq} \mathsf{Fun}^{\otimes}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$$

between subcategories of (strongly) symmetric monoidal functors.

Proof. Since $y^\otimes$ is (strongly) symmetric monoidal, the functor $- \circ y^\otimes$ preserves strong monoidality. On the other hand, if $F \in \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$ is actually strongly symmetric monoidal (which, once we have specified a lax monoidal structure, is a property and does not require additional data), let $H = \mathsf{LKE}^\otimes(F)$. Then, we have to show that H is strongly symmetric monoidal, i.e., that for any $\mathcal{F},\mathcal{G} \in \mathcal{P}(\mathcal{C})$, the map $H(\mathcal{F}) \otimes_{\mathcal{D}} H(\mathcal{G}) \longrightarrow H(\mathcal{F} \otimes_{\mathcal{P}(\mathcal{C})} \mathcal{G})$ (coming from the lax symmetric monoidal structure on H) is an equivalence. By assumption, this holds for $\mathcal{F}$ and $\mathcal{G}$ representable presheaves. Moreover, the class of presheaves $\mathcal{F}$ and $\mathcal{G}$ for which this holds is closed under colimits in both $\mathcal{F}$ and $\mathcal{G}$. Indeed, H preserves

colimits by construction, and tensor products in both $\mathcal{D}$ and $\mathcal{P}(\mathcal{C})$ preserve colimits in each variable (by Proposition A.2). Since every presheaf is a colimit of representable ones, this concludes the proof. □

Appendix B. Slice ∞-categories

For $\mathcal{C}$ a category and $X \in \mathcal{C}$, one defines a *slice category* $\mathcal{C}_{/X} := \mathsf{Fun}(\Delta^1, \mathcal{C}) \times_{\mathsf{Fun}(\{1\},\mathcal{C})} \{X\}$ [Lur09, Prop. 1.2.9.2]. This can be thought of as the category of edges in $\mathcal{C}$ with target X, and morphisms over X. It comes with a forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$, remembering the source of an edge. Slice categories are an essential ingredient of our discussion in Section 2. Here are some results about them that we have used.

Definition B.1. Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{C}$ a subcategory. Then the category $\mathcal{D}_{/X}$ is defined as the fiber of the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ over $\mathcal{D}$.

Proposition B.2. *Let $\mathcal{C}$ be a category, and $\mathcal{F} \in \mathcal{P}(\mathcal{C})$. We view $\mathcal{C}$ as a subcategory of $\mathcal{P}(\mathcal{C})$ via the Yoneda embedding. Then, the left Kan extension of the embedding $\mathcal{C}_{/\mathcal{F}} \to \mathcal{P}(\mathcal{C})_{/\mathcal{F}}$ (from Definition B.1) defines an equivalence of categories*

$$\theta : \mathcal{P}(\mathcal{C}_{/\mathcal{F}}) \xrightarrow{\simeq} \mathcal{P}(\mathcal{C})_{/\mathcal{F}}.$$

The functor θ restricts to an equivalence

$$\mathcal{P}_\Sigma(\mathcal{C}_{/\mathcal{F}}) \xrightarrow{\simeq} \mathcal{P}_\Sigma(\mathcal{C})_{/\mathcal{F}}.$$

Proof. The first claim is proven in [ABG18, §5.3]. For the second part of the statement, consider $\mathcal{G} \in \mathcal{P}_\Sigma(\mathcal{C})_{/\mathcal{F}}$. Then $\mathcal{G}$ is a sifted colimit of arrows of the form $y(c) \to \mathcal{F}$ for $c \in \mathcal{C}$, so that $\theta^{-1}(\mathcal{G})$ is a sifted colimit of representables $y(c \to \mathcal{F})$. It is thus a spherical presheaf in $\mathcal{P}(\mathcal{C}_{/\mathcal{D}})$. The argument for the converse is the same. □

Slices of symmetric monoidal categories over commutative algebra objects inherit a symmetric monoidal structure:

Proposition B.3 ([Lur17, Thm. 2.2.2.4 and Rmk. 2.2.2.5])**.** *Let $\mathcal{D}^\otimes$ be a symmetric monoidal category and $X \in \mathsf{CAlg}(\mathcal{D})$. Then the slice $\mathcal{D}_{/X}$ admits a symmetric monoidal structure $\mathcal{D}^\otimes_{/X}$, making in particular the projection $\mathcal{D}^\otimes_{/X} \to \mathcal{D}^\otimes$ symmetric monoidal. Informally, the tensor product of two objects $c \to X$ and $d \to X$ in the slice can be described by*

$$(c \to X) \otimes (d \to X) = (c \otimes d \to X \otimes X \to X)$$

where $X \otimes X \to X$ is given by the multiplicative structure of X.

As a particular case of Proposition B.2, for $\mathcal{C} = *$, so that $\mathcal{P}(\mathcal{C}) \simeq \mathsf{Spc}$, we obtain the following result.

Lemma B.4 ([ABG18, §6.2])**.** *Let $X \in \mathsf{CMon}(\mathsf{Spc})$. In the case $\mathcal{C} = *$, Proposition B.2 provides an equivalence of symmetric monoidal categories*

$$\mathsf{Spc}_{/X} \simeq \mathcal{P}(X),$$

where $\mathsf{Spc}_{/X}$ is endowed with the symmetric monoidal structure from Proposition B.3, and $\mathcal{P}(X)$ is the category of presheaves on X with the Day convolution symmetric monoidal structure (see Subsection A), viewing X as a symmetric monoidal category.

The symmetric monoidal structure from Proposition B.2 has a very convenient universal property:

Proposition B.5 ([ACB19, Lem. 2.12])**.** *Let $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ be symmetric monoidal categories classified by coCartesian fibrations $p : \mathcal{C}^\otimes \to \mathsf{Fin}_*$ and $q : \mathcal{D}^\otimes \to \mathsf{Fin}_*$. Let $X \in \mathsf{CAlg}(\mathcal{D})$, and $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a lax symmetric monoidal functor.*

Then lax symmetric monoidal functors $\mathcal{C}^\otimes \to \mathcal{D}^\otimes_{/X}$ lifting F correspond to symmetric monoidal natural transformations $F \to X \circ p$, where X is viewed as a section of q. More precisely, there is an equivalence

$$\mathsf{Map}_{\mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)}(F, X \circ p) \simeq \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes, \mathcal{D}^\otimes_{/X}) \times_{\mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)} \{F\}.$$

In particular, for $\mathcal{C}^\otimes = \mathcal{E}_n^\otimes$, viewing F as an $\mathcal{E}_n$-algebra $Y \in \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^\otimes)$, then $\mathcal{E}_n$-algebras in $\mathcal{D}^\otimes_{/X}$ lifting $Y \in \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^\otimes)$ correspond to morphisms of $\mathcal{E}_n$-algebras $Y \to X$ in $\mathcal{D}^\otimes$. More precisely, there is an equivalence

$$\mathsf{Map}_{\mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^\otimes)}(Y, X) \simeq \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^\otimes_{/X}) \times_{\mathsf{Alg}_{\mathcal{E}_n}(\mathcal{D}^\otimes)} \{Y\}.$$

Remark B.6. Both Propositions B.3 and B.5 are proven in a more general setting in the references we gave; namely for $\mathcal{O}^\otimes$-monoidal categories, for some ∞-operad $\mathcal{O}^\otimes$. We will only need the case $\mathcal{O}^\otimes = \mathsf{Fin}_*$.

Index of notation

$\mathbb{A}^1$	Affine line over some base scheme	
$\mathsf{Alg}(\mathcal{C}^\otimes)$, $\mathsf{CAlg}(\mathcal{C}^\otimes)$	Categories of $\mathcal{E}_1$-, resp. $\mathcal{E}_\infty$-algebras in a symmetric monoidal category $\mathcal{C}^\otimes$	§1.3
$\mathcal{C}^\simeq$	Maximal (∞-)groupoid in a category $\mathcal{C}$ (right adjoint to the forgetful functor)	2.1
Cat_∞	Category of small (∞-)categories	
η	Motivic Hopf map	§3.2
$\mathcal{E}_n$	∞-operad of little n-cubes	§1.3
f_n, $\tilde{f}_n$	n-effective, respectively very n-effective cover functors	1.7
Fin	1-category of finite sets (including $\emptyset$)	2.14
$\mathsf{Fun}^\times(\mathcal{C},\mathcal{D})$	Category of functors $\mathcal{C}\to\mathcal{D}$ preserving finite products	2.15
$\mathsf{Fun}^\mathsf{L}(\mathcal{C},\mathcal{D})$	Category of colimit-preserving functors $\mathcal{C}\to\mathcal{D}$	A.3
$\mathsf{Fun}^\otimes(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$	Category of symmetric monoidal functors $\mathcal{C}^\otimes\to\mathcal{D}^\otimes$	A.5
$\mathsf{Fun}^\mathsf{lax}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$	Category of lax symmetric monoidal functors $\mathcal{C}^\otimes\to\mathcal{D}^\otimes$	A.1
$\mathbb{G}_m$	Group scheme $S\times_{\mathsf{Spec}(\mathbb{Z})}\mathsf{Spec}(\mathbb{Z}[t,t^{-1}])$ over a base scheme S	
$\mathsf{HA}[t^n]$	Free $\mathcal{E}_1$-HA-algebra on one generator in degree n	1.20
$\mathsf{H}\mathbb{Z}$, $\mathsf{H}\mathbb{Z}/2$	Motivic Eilenberg-Mac Lane spectra, representing motivic cohomology	3.1
$\widetilde{\mathsf{H}\mathbb{Z}}$	Effective cover of the Milnor-Witt K-theory sheaf: $f_0\underline{K}^{MW}_*$	§3.3
$\underline{K}^{MW}_*$, $\underline{K}^M_*$, $\underline{K}^W_*$	Milnor-Witt, Milnor, and Witt K-theory sheaves	3.1
KGL, kgl	Algebraic K-theory motivic spectrum and its very effective cover	3.2
KO, ko	Hermitian K-theory motivic spectrum and its very effective cover	3.4
KO^top, ko^top	Real topological K-theory spectrum and its connective cover	3.4
KU	Complex topological K-theory spectrum	4.6
KW, kw	Balmer-Witt K-theory motivic spectrum and its very effective cover	3.12
$\mathsf{L}(-)$	L-theory spectrum functor	3.12
M	Motivic Thom spectrum functor	2.25
$\mathsf{Map}_\mathcal{C}$	Mapping space in a category $\mathcal{C}$	
MGL, MSL, MSp	General linear, special linear, and symplectic cobordism motivic Thom spectra	2.27
MO, MSO, MU	Unoriented, oriented, and complex cobordism Thom spectra	2.5
M_top	Topological Thom spectrum functor	2.1
$\langle n\rangle$, $\langle n\rangle^\circ$	Finite pointed set $\{*,1,\dots,n\}$ in Fin_*, resp. finite set $\{1,\dots,n\}$ in Fin	2.14
$\mathcal{C}^\omega$	Full subcategory spanned by compact objects in a category $\mathcal{C}$	2.1
$\Omega^\infty,\Omega^\infty_T$	Infinite loop space functors for classical, resp. motivic ($\mathbb{P}^1$-)spectra	
$\underline{\pi}_*(-)_*$, $\underline{\pi}_{*,*}(-)$	Homotopy sheaves of a motivic space or spectrum	
$\mathcal{P}(\mathcal{C})$	Category of presheaves of spaces on a category $\mathcal{C}$	
$\mathcal{P}_\Sigma(\mathcal{C})$	Category of spherical presheaves of spaces on a category $\mathcal{C}$	2.12
Pr^L	Category of presentable categories and left-adjoint functors	
ρ	Inclusion $\mathcal{S}^0\to\mathbb{G}_m$ on the points corresponding to 1 and -1	1.13
$r_\mathbb{R}$, $r_\mathbb{C}$	Real and complex Betti realization functors	§1.2
Spc	Category of spaces	
$\mathcal{S}$	Motivic sphere spectrum	
$\mathcal{S}^n$	(Simplicial) n-sphere in the category of motivic spaces or motivic spectra	
$\mathcal{S}^{p,q}$	Motivic bigraded spheres, $\mathcal{S}^{p,q}=\mathcal{S}^{p-q}\wedge\mathbb{G}_m^{\wedge q}$	
s_n, $\tilde{s}_n$	n-effective, respectively very n-effective slices	1.7
$\mathbb{S}$	Topological sphere spectrum	
$\mathbb{S}^n$	n-sphere in the category of spaces or spectra	
$\mathsf{SH}(S)$	Category of motivic ($\mathbb{P}^1$-)spectra over S	
$\mathsf{SH}(S)^\mathsf{eff}$, $\mathsf{SH}(S)^\mathsf{veff}$	Subcategories of effective, respectively very effective spectra over S	1.1
$\mathsf{SH}(S)^\mathsf{eff}(n)$, $\mathsf{SH}(S)^\mathsf{veff}(n)$	Subcategories of n-effective, respectively very n-effective spectra over S	1.1
Σ_T	Functor $T\wedge-:\mathsf{SH}(S)\to\mathsf{SH}(S)$	
Σ^∞	Infinite suspension functors for topological or motivic spectra	
Sm_S	1-category of smooth schemes of finite type over the base scheme S	
Sp	Category of (topological) spectra	
Span	When working over a base scheme S, $\mathsf{Span}(\mathsf{Sm}_S,\mathsf{all},\mathsf{fold})$	2.17
$\mathsf{Span}(\mathcal{C},\mathsf{left},\mathsf{right})$	Category of objects in $\mathcal{C}$, with morphisms certain span diagrams	2.13
$\mathsf{Spc}(S)$	Category of motivic spaces over the base scheme S	
T	Pointed motivic space or spectrum corresponding to $(\mathbb{P}^1,\infty)$	
$\tau_{\geq n},\tau_{\leq n}$	Truncation functors for a t-structure	
$(-)_+$	Disjoint base point functor (free functor from unpointed to pointed objects)	

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Institut für Mathematik, JGU Mainz, Germany
Email address: bannwart.julie@gmail.com