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Model categories and homotopy: the example of topological spaces and simplicial sets

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1 Introduction

Topological spaces can be studied from the point of view of homotopy theory, which allows us to compare maps and spaces, to classify some of their properties that are "invariant under continuous deformation" and much more. In topology, homotopy is an equivalence relation between maps. Homotopy equivalences and weak homotopy equivalences are particular types of maps, that are not homeomorphisms, but preserve much structure. The first ones have an inverse up to homotopy.

Interestingly, these constructions can be generalized and axiomatized in order to enlarge the notion of homotopy to other structures, such as chain complexes of modules over a ring and simplicial sets. It turns out that the structure of model category, introduced by Quillen (1967), is a suitable setting for this construction. A model category is a category with three distinguished classes of morphisms: *weak equivalences*, *fibrations* and *cofibrations*, satisfying certain axioms noticeably similar to some properties of particular classes of maps between topological spaces. We will make this definition precise in section (2). The weak equivalences generalize the notion of homotopy equivalences, or weak homotopy equivalences. In this setting, we can define homotopy of maps, which will be an equivalence relation only under some assumptions. By considering equivalence classes for this relation, we obtain a new category, the *homotopy category*, where weak equivalences correspond, in a sense yet to be defined, to invertible maps. This is another way in which the homotopy category can be constructed: by formally inverting the weak equivalences. This uses the categorical notion of *localization*. Both constructions will be described in section (3).

Since we advertised the concept of model category as generalizing topological notions, we start by considering the *Quillen model structure* on the category of topological spaces **Top** (section (5)), where the weak equivalences will be the weak homotopy equivalences.

Then, we continue with another example of model structure: the *Quillen model structure* on the category of simplicial sets $\mathbf{sSet}$ (introduced in section (6)). Simplicial sets are rather combinatorial objects, formally defined as a certain category of functors, but intuitively they can be seen as a way to represent topological spaces that can be constructed from *standard simplices*: points, segments, triangles, tetrahedra, and so on. Each simplicial set has a topological space associated to it, called its *geometric realization*, and this space is built as a quotient of a disjoint union of such standard simplices. This construction is functorial, and the weak equivalences for the model structure on the category of simplicial sets will be chosen as the maps whose geometric realization is a (topological) weak homotopy equivalence.

Quite surprisingly, simplicial sets turn out to be good approximations of topological spaces up to weak homotopy equivalence: we will show in section (7) the following theorem:

Theorem 7.5. *There exists an adjunction*

$$|\cdot| : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : S$$

between the categories of simplicial sets and topological spaces, both with the Quillen model structure, that is also a Quillen equivalence, namely there is an adjunction and an equivalence of categories between their respective homotopy categories $\mathrm{Ho}(\mathbf{Top})$ *and* $\mathrm{Ho}(\mathbf{sSet})$*.*

This result is a particular case of a more general theorem allowing us to turn an adjunction between model categories, into an adjunction or even an equivalence in some cases, between their homotopy categories: we will prove in section (4) the theorem:

Theorem 4.4. *Let C and D be model categories with an adjunction:*

$$F : C \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} D : G$$

If one of the following holds:

(i) The left adjoint F preserves cofibrations and acyclic cofibrations (maps that are both cofibrations and weak equivalences)

(ii) The right adjoint G preserves fibrations and acyclic fibrations (maps that are both fibrations and weak equivalences)

(iii) F preserves cofibrations and G preserves fibrations

then there exists an adjoint pair of functors:

$$\mathbb{L}F : \mathrm{Ho}(C) \underset{\perp}{\rightleftarrows} \mathrm{Ho}(D) : \mathbb{R}G$$

between their respective homotopy categories.

Under the additional assumption that the natural bijections

$$\alpha_{X,Y} : D(F(X), Y) \to C(X, G(Y)) \quad \text{and} \quad \beta_{X,Y} : C(X, G(Y)) \to D(F(X), Y)$$

induced by the adjunction $F \dashv G$ both preserve weak equivalences for every cofibrant object X in C and fibrant object Y in D, the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ is an equivalence of categories $\mathrm{Ho}(C) \simeq \mathrm{Ho}(D)$.

(we will see more precisely what fibrant and cofibrant objects are in section (2))

Finally, in section (8), to illustrate the concepts we studied, we will explore other examples of model structures, especially on categories of diagrams. This will be a more descriptive part, without many proofs. Algebraic theories are categories providing us with a way of encoding the axioms defining certain algebraic structures. Each such structure will then be represented by a *strict algebra*, a product preserving functor from the suitable algebraic theory to the category of simplicial sets. When the products are preserved up to weak equivalence only, this functor is called a *homotopy algebra*. We will study categories that can represent both types of algebras (Alg_T for strict algebras and $\mathrm{LFun}(T, \mathbf{sSet})$ for homotopy algebras over an algebraic theory T) and model structures on them (several ones on the same category). Another application of theorem (4.4) shows the following:

Theorem 8.9. *There is a Quillen equivalence:*

$$K_T : \mathrm{LFun}(T, \mathbf{sSet}) \underset{\perp}{\rightleftarrows} \mathrm{Alg}_T : J_T.$$

We will also see that we can "strengthen" any homotopy algebra into a strict algebra, up to a suitable notion of weak equivalence:

Theorem 8.10. *Let T be an algebraic theory. Every homotopy T-algebra $F : T \to \mathbf{sSet}$ is weakly equivalent to a strict T-algebra $F' : T \to \mathbf{sSet}$, i.e. there exists a natural transformation $\tau : F \to F'$ such that τ_t is a weak equivalence of simplicial sets for all objects t in T.*

For sections (2) to (5) we follow the article "Homotopy theories and model categories" by Dwyer and Spaliński (1995) (see the bibliography (9) at the end for more details), and for the model structure on $\mathbf{sSet}$ we rely on the definitions and ideas by Goerss and Jardine (1999) in their book "Simplicial homotopy theory" and also lecture notes about homotopy theory by Jardine (2018). The last part is based on the article "Algebraic theories in homotopy theory" by Badzioch (2002). While discussing all these notions, we will extensively use common categorical constructions and their properties, especially limits and colimits. For a brief discussion of this basic concepts, see Section 2 in the article by Dwyer and Spaliński (1995).

Conventions

We will use the following notational conventions:

- For a category $C = (\mathrm{Ob}\ C, \mathrm{Mor}\ C)$ and two objects $a, b \in \mathrm{Ob}\ C$, $C(a, b)$ denotes the class of morphisms $f \in \mathrm{Mor}\ C$ with domain a and codomain b. We write $f \in C(a, b)$ or $f : a \to b$. The words *morphisms*, *arrows* and *maps* will be used interchangeably to talk about the elements of $\mathrm{Mor}\ C$. We will sometimes omit the composition symbol $\circ$: the composition of $f \in C(a, b)$ with $g \in C(b, c)$ is denoted by $g \circ f$ or gf. The map id_a is the identity morphism of an object a of C.
- In diagrams, double bars like $x \Longleftarrow\!\!\!\!= x$ stand for the identity.

- For C a category, $C^{\to}$ is the category of morphisms of C (with objects $\mathrm{Mor}\, C$ and for $f \in C(a,b)$, $g \in C(c,d)$, we have $C^{\to}(f,g) = \{(h,k) \in C(a,c) \times C(b,d) \mid gh = kf\}$, i.e. morphisms are commutative squares in C).
- Corner symbols $\lrcorner$ and $\ulcorner$ denote pushouts and pullbacks respectively. For some category C, the colimit of a functor $N \to C$ where N is the category associated to the poset $(\mathbb{N}, \leq)$ will be called a *sequential colimit*.
- When writing a diagram, we mean that it is commutative.
- When working with an equivalence relation, $[x]$ denotes the equivalence class of an element x for the relation under consideration.
- If X, Y, Z are objects in a category, with $X \amalg Y$ a coproduct of X and Y, $f + g : X \amalg Y \to Z$ denotes the map given by the universal property of coproducts applied to some maps $f : X \to Z$ and $g : Y \to Z$. If $X \times Y$ is a product of X and Y, $(h,k) : Z \to X \times Y$ denotes the map obtained by the universal property of products applied to the maps $h : Z \to X$ and $k : Z \to Y$. If Z itself is the product $A \times B$ with projections $\pi_0 : Z \to A$ and $\pi_1 : Z \to B$, given maps $\ell : A \to X$ and $m : B \to Y$, we denote by $\ell \times m : A \times B \to X \times Y$ the map $(\ell \circ \pi_0, m \circ \pi_1)$.
- Standard topological spaces: I denotes the interval $[0,1]$ in $\mathbb{R}$, for any $n \in \mathbb{N}$, $\mathbb{D}^n$ denotes the n-disk $\{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1\}$ and $\mathbb{S}^{n-1}$ the $(n-1)$-sphere $\{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = 1\}$, all with the Euclidean topology ($\mathbb{D}^0$ is a point, and $\mathbb{S}^{-1} = \emptyset$). For the n-sphere, we choose the base point $(1, 0, \ldots, 0) = s_0$.

2 Model categories

2.1 The definition and axioms

Before stating the main definition, we introduce the notion of retracts. In topology, if X is a space and A a subspace, we say that A is a retract of X if there exists a continuous map $f : X \to A$ such that $f|_A = \mathrm{id}_A$, namely if $f \circ \iota = \mathrm{id}_A$ where $\iota : A \to X$ is the inclusion. Here is the categorical version:

Definition 2.1 (Retract)**.** Let C be a category. An object a is called a *retract* of another object x if there exists morphisms $\iota : a \to x$ and $r : x \to a$ such that $r\iota = \mathrm{id}_a$. A map $f : a \to b$ is a called a *retract* of another map $g : c \to d$ if f is a retract of g in the preceding sense, where f and g are seen as objects in the category $C^{\to}$, i.e. if there exists in C a diagram of the form:

$$\begin{tikzcd} c \arrow[r, shift right, "r_a"'] \arrow[d, "g"'] & a \arrow[l, shift right, "\iota_a"'] \arrow[d, "f"] \\ d \arrow[r, shift right, "r_b"'] & b \arrow[l, shift right, "\iota_b"'] \end{tikzcd}$$

with $r_a\iota_a = \mathrm{id}_a$, $r_b\iota_b = \mathrm{id}_b$ (i.e. a is a retract of c and b is a retract of d, in a compatible way with f and g). This diagram is also often drawn as:

$$\begin{tikzcd} a \arrow[r, "\iota_a"] \arrow[d, "f"'] & c \arrow[r, "r_a"] \arrow[d, "g"] & a \arrow[d, "f"] \\ b \arrow[r, "\iota_b"] & d \arrow[r, "r_b"] & b \end{tikzcd}$$

Now we can give the main definition. The definition of a model category changes a bit depending on the authors. The definition given by Dwyer and Spaliński (1995) is not far from what Quillen (1967) called a *closed* model category, and requires the existence of *finite* limits and colimits only. However, other definitions ask for the existence of all *small* (co)limits. Since all the examples we will work with satisfy this stronger property, we choose the second version here. Moreover, some authors ask for *functorial* factorizations in the last axiom (like Hovey (1999) for example).

Definition 2.2 (Model category)**.** Let C be a category. We call a *model category structure* on C the data of three classes of morphisms of C:

(i) $\mathbf{We}(C)$, the class of *weak equivalences* (if f is a weak equivalence we write $f : a \xrightarrow{\sim} b$)

(ii) $\mathbf{Cofib}(C)$, the class of *cofibrations* (if f' is a cofibration we write $f' : a' \hookrightarrow b'$)

(iii) $\mathbf{Fib}(C)$, the class of *fibrations* (if f'' is a fibration we write $f'' : a'' \twoheadrightarrow b''$)

satisfying the following axioms:

MC0 (Stability and identities:) The three classes of morphisms above are stable by composition and contain all identity morphisms.

MC1 (Limits and colimits:) The category C has all small limits and colimits.

MC2 ("2 of 3" rule:) If $f : a \to b$, $g : b \to c$ are morphisms in C, and two morphisms among f, g and gf are weak equivalences, then so is the third.

MC3 (Retracts:) A retract of a cofibration/fibration/weak equivalence is a cofibration/fibration/weak equivalence respectively.

MC4 (Lifts:) Consider a diagram in C:

$$\begin{tikzcd} a \arrow[r] \arrow[d, hook, "\iota"'] & c \arrow[d, two heads, "p"] \\ b \arrow[r] & d \end{tikzcd}$$

If either ι or p is a weak equivalence, then a *lift* ℓ exists in the diagram above, i.e. there exists a morphism $\ell : b \to c$ such that the following diagram is commutative:

$$\begin{tikzcd} a \arrow[r] \arrow[d, hook, "\iota"'] & c \arrow[d, two heads, "p"] \\ b \arrow[r] \arrow[ur, dashed, "\ell"] & d \end{tikzcd}$$

We do not require this lift to be unique.

MC5 (Factorization:) Any map $f : a \to b$ in C has two factorizations:

$$\begin{tikzcd} a \arrow[r, hook, "\iota(f)", "\sim"'] \arrow[dr, "f"'] & \omega(f) \arrow[d, two heads, "\pi(f)"] \\ & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, hook, "\iota'(f)"] \arrow[dr, "f"'] & \omega'(f) \arrow[d, two heads, "\pi'(f)", "\wr"'] \\ & b \end{tikzcd}$$

If we call an *acyclic fibration* a map that is both a fibration and a weak equivalence and an *acyclic cofibration* a map that is both a cofibration and a weak equivalence, the axiom states that any map has a first factorization as the composition of an acyclic cofibration and a fibration, and a second factorization as the composition of a cofibration and an acyclic fibration (for any fixed map f in C, $\omega(f)$ and $\omega'(f)$ are fixed objects, and $\iota(f), \pi(f), \iota'(f), \pi'(f)$ fixed maps: this axiom provides the existence of such factorizations, but also chooses two for each map f in C).

Axiom MC5 is stated a bit differently here than in the paper by Dwyer and Spaliński (1995): having a choice of factorizations for each map will allow us to define the functor R in section (3) when the collection of objects of C is not necessarily a set but might be a proper class. However, during most of the proofs we will do, only the existence of such factorizations given a map f matters, so we will forget about this formalism and simply consider some maps π and ι in C with the desired property.

Furthermore, we restrict ourselves to *locally small* categories, i.e. such that the collection of morphisms between two fixed objects is a set and not a proper class (because later we will study equivalence relations on these sets, and applications between sets of maps between fixed objects for example).

Definition 2.3 (Right/left lifting property)**.** In the situation of *MC4*, when a lift exists in the diagram, we say that p has the *right lifting property (RLP)* with respect to ι, and that ι has the *left lifting property (LLP)* with respect to p. This terminology applies whenever we have a commutative square with a lift, even if the maps involved are not (co)fibrations.

Definition (2.2) actually applies in a wide variety of contexts. For instance:

Example 2.4. The category of chain complexes of left-modules over a given ring R can be endowed with a model category structure. We will not discuss this example further, but some details are given in Section 7 in the article by Dwyer and Spaliński (1995).

Example 2.5. The category **Top** can be endowed with at least two different model structures. Firstly, the following:

(i) **We**(**Top**) defined as the class of homotopy equivalences of topological spaces (in the usual sense)

(ii) **Fib**(**Top**) defined as the class of *Hurewicz fibrations*, that is, maps with the homotopy lifting property, i.e. RLP with respect to inclusions $A \times \{0\} \to A \times [0, 1]$ for any topological space A

(iii) **Cofib**(**Top**) defined as the class of maps having the LLP with respect to all maps that are both Hurewicz fibrations and homotopy equivalences

define a model structure on **Top**, due originally to Strøm (1972). Again we will not prove this, but we will prove in section (5) the existence of another model structure on **Top**, where the weak equivalences are the weak homotopy equivalences in the usual topological sense.

Remark 2.6. We already notice some connection between the axioms in definition 2.2 and some properties of topological spaces: the existence of lifts MC4 recalls important properties of coverings, retracts are commonly used in topology, MC2 is verified for homotopy equivalences because they admit a homotopy inverse, and so on. In fact our construction of the homotopy category in section 3 will generalize this last property to any model category: the weak equivalences will become isomorphisms in the homotopy category, i.e. maps with a "homotopy inverse".

2.2 General properties

Let C be a model category.

Axiom MC1 implies that the empty diagram has a colimit and a limit in C, namely there exists an initial object $\emptyset$ and a terminal object $*$ in C, such that for any object X in C there exists a unique map $X \to *$ and a unique map $\emptyset \to X$.

Definition 2.7 ((Co)fibrant)**.** Let X be an object in C. We say that X is *cofibrant* if the unique map $\emptyset \to X$ is a cofibration, and that X is *fibrant* if the unique map $X \to *$ is a fibration. We call X *bifibrant* if it is both fibrant and cofibrant.

The propositions (3.26) and (3.27) in section (3) will prove in particular that such objects exist.

2.2.1 The notion of duality

We will use extensively a notion of duality between fibrations and cofibrations: the axioms in definition (2.2) remain unchanged if we reverse the direction of all arrows and replace "fibration" by "cofibration" and vice-versa. In other words, given a model category C, call a map f^{op} in C^{op} a weak equivalence, respectively a fibration/cofibration if and only if f in C is a weak equivalence, respectively a cofibration/fibration. Then this choice endows C^{op} with a model category structure. For instance, note for MC1 that limits in C correspond to colimits in C^{op} and vice versa, and for MC3 that an object b is a retract of a in C with maps $b \xrightarrow{\iota} a \xrightarrow{r} b$ if and only if b is a retract of a in C^{op} with maps $b \xrightarrow{r^{\mathrm{op}}} a \xrightarrow{\iota^{\mathrm{op}}} b$ (because $\iota^{\mathrm{op}} r^{\mathrm{op}} = (r\iota)^{\mathrm{op}} = \mathrm{id}_a^{\mathrm{op}}$), and that the diagrams in the other axioms are symmetric under the "dualization" described above.

Therefore, if we prove a general statement about model categories, then the dual statement obtained by exchanging fibrations and cofibrations and reversing directions of arrows is also true.

Example 2.8. Since duality exchanges limits and colimits, fibrations and cofibrations, the dual concept to a cofibrant object a (the map from the colimit $\emptyset$ of the empty diagram to a is a cofibration) is the concept of a fibrant object (the map from a to the limit $*$ of the empty diagram is a fibration).

2.2.2 Lifting properties

Axiom MC4 requires lifting properties for certain types of maps. Actually these lifting properties uniquely determine these maps:

Proposition 2.9 (A converse to MC4)**.** *The class of cofibrations in C is exactly the class of maps having the LLP with respect to all acyclic fibrations, and the class of acyclic cofibrations is exactly the class of maps having the LLP with respect to all fibrations. Dually, the class of fibrations is exactly the class of maps having the RLP with respect to all acyclic cofibrations, and the class of acyclic fibrations is exactly the class of maps having the RLP with respect to all cofibrations.*

Proof. By duality it suffices to prove the two first statements. By the lift axiom MC4, cofibrations, respectively acyclic cofibrations, have the desired LLP with respect to acyclic fibrations, respectively fibrations. Now suppose that $f : X \to Y$ is a map in C having the LLP with respect to all acyclic fibrations. By the factorization axiom MC5, there exist maps $p : X' \xrightarrow{\sim} Y$ and $\iota : X \hookrightarrow X'$ with $p\iota = f$. By hypothesis, a lift exists in the following diagram:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d, "f"'] & X' \arrow[d, two heads, "p", "\wr"'] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\exists \ell"] & Y \end{tikzcd}$$

This allows us to express f as a retract of ι:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, hook, "\iota"] & X \arrow[d, "f"] \\ Y \arrow[r, "\ell"] \arrow[rr, equal, bend right] & X' \arrow[r, "p"] & Y \end{tikzcd}$$

Then, by the retract axiom MC3, f is a cofibration. For the second statement, if f has the LLP with respect to all fibrations, we repeat the proof above, using the other part of the factorization axiom, with ι being a weak equivalence and not necessarily p. A lift exists in the same diagram as above, and it expresses f as a retract of the acyclic cofibration ι, so by MC3 again, f is a cofibration and a weak equivalence, that is, an acyclic cofibration. □

In any category, lifting properties are stable under some common categorical operations:

Proposition 2.10 (Stability of lifting properties). *Consider any category D (not necessarily a model category) and a class of maps S in D. Let L be the class of maps of D having the LLP with respect to all maps in S. Then:*

(i) L contains all identities

(ii) if $f, g \in L$ with $f \in D(a,b)$, $g \in D(b,c)$ then $g \circ f \in L$

(iii) in a pushout diagram, the cobase change of a map in L is again in L

(iv) if $g \in L$ and f is a retract of g, then $f \in L$

(v) if the sequential colimit x of the diagram $x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \dots$ in D exists, with $f_n \in L$ for all $n \in \mathbb{N}$, then the induced map $x_0 \to x$ is again in L.

In other words, L is stable under composition, cobase change, retracts and sequential colimits. The same statements are true for RLP, for composition, base change in pullback squares, and retracts.

Proof. We only do the proof for the LLP. For the RLP the proof is symmetric.

(i) Let $s \in D(d,c) \cap S$, and consider id_a for some object a of D. Given maps $k \in D(a,d)$ and $h \in D(a,c)$ with $sk = h$, we want to find a lift in:

$$\begin{tikzcd} a \arrow[r, "k"] \arrow[d, equal] & d \arrow[d, "s"] \\ a \arrow[r, "h"'] & c \end{tikzcd}$$

The map k satisfies the necessary conditions. By arbitrarity of s, k and h, we get $\mathrm{id}_a \in L$.

(ii) Let $f, g \in L$ with $f \in D(a,b)$, $g \in D(b,c)$ as in the statement. Let $s \in S$. The first diagram shows a lifting problem:

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] & d \arrow[dd, "s"] \\ b \arrow[d, "g"'] & \\ c \arrow[r, "k"'] & e \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] & d \arrow[dd, "s"] \\ b \arrow[d, "g"'] \arrow[ur, dotted, "\ell"] & \\ c \arrow[r, "k"'] \arrow[uur, dotted, "\ell'"'] & e \end{tikzcd}$$

We build a lift in two steps as shown in the second diagram. Since $f \in L$, a lift exists in the square formed by the maps h, s, kg and f: there exists a map $\ell : b \to d$ such that $s\ell = kg$ and $\ell f = h$. Since $g \in L$, a lift exists in the square formed by the maps ℓ, s, k and g: there exists $\ell' : c \to d$ such that $s\ell' = k$ and $\ell' g = \ell$. Finally ℓ' is the desired lift: $s\ell' = k$ and $\ell' gf = \ell f = h$ by construction. Hence gf has the LLP with respect to s. By arbitrarity of s, we have $gf \in L$.

(iii) Suppose that $c \xrightarrow{j} d \xleftarrow{i} b$ is a pushout of $c \xleftarrow{f} a \xrightarrow{g} b$ in D. Suppose that $f \in L$. We want to show that the cobase change of f along g, namely i, is an element of L. Let $s \in S \cap D(x,y)$ and suppose we want to find a lift in the square:

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[d, "i"'] & x \arrow[d, "s"] \\ d \arrow[r, "k"'] & y \end{tikzcd}$$

We use the lifting property of f to find a lift ℓ in the rectangle formed by hg, s, kj and f. Then the universal property of the pushout applied to the maps $h : b \to x$ and $\ell : c \to x$ induces a map $\ell' : d \to x$. We have the following situation:

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"'] & b \arrow[r, "h"] \arrow[d, "i"] & x \arrow[d, "s"] \\ c \arrow[r, "j"'] \arrow[urr, dashed, "\ell"] & d \arrow[r, "k"'] \arrow[ur, dashed, "\ell'"'] \arrow[ul, phantom, "\ulcorner"] & y \end{tikzcd}$$

And we check that ℓ' is the desired lift: $s\ell'$ is equal to k because both maps satisfy the universal property of the pushout for the maps kj and ki (since $s\ell' j = s\ell = kj$ and $s\ell' i = sh = ki$ by construction), and $\ell' i = h$ because ℓ' was induced by h. We conclude that $i \in L$.

(iv) Consider a diagram expressing f as a retract of $g \in L$:

$$\begin{tikzcd} a \arrow[r, "\iota_a"] \arrow[d, "f"'] & c \arrow[r, "r_a"] \arrow[d, "g"] & a \arrow[d, "f"] \\ b \arrow[r, "\iota_b"] & d \arrow[r, "r_b"] & b \end{tikzcd}$$

Let $s \in S \cap D(x, y)$. We look for a lift in the diagram:

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] & x \arrow[d, "s"] \\ b \arrow[r, "k"'] & y \end{tikzcd}$$

By hypothesis, a lift exists in the square formed by maps hr_a, s, kr_b and g: there is a map $\ell : d \to x$ such that $s\ell = kr_b$ and $\ell g = hr_a$. Then, $\ell\iota_b : b \to x$ is the desired lift: indeed, $s\ell\iota_b = kr_b\iota_b = k$ and $\ell\iota_b f = \ell g\iota_a = hr_a\iota_a = h$ by hypothesis.

(v) Now consider a diagram as in the statement, which admits a sequential colimit:

$$\begin{tikzcd} x_0 \arrow[r, "f_0"] \arrow[drr, "i_0"'] & x_1 \arrow[r, "f_1"] \arrow[dr, "i_1"] & x_2 \arrow[r, "f_2"] \arrow[d, "i_2"] & \dots \\ & & x = \operatorname{colim} x_i & \end{tikzcd}$$

Suppose $f_n \in L \ \forall n \in \mathbb{N}$. Let $s \in S \cap D(a, b)$ and consider a lifting problem:

$$\begin{tikzcd} x_0 \arrow[r, "h"] \arrow[d, "i_0"'] & a \arrow[d, "s"] \\ x \arrow[r, "k"'] & b \end{tikzcd}$$

For all $n \in \mathbb{N}^*$, we can inductively define a map j_n from x_n to a by finding a lift in the square (set $j_0 = h$):

$$\begin{tikzcd} x_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, "f_{n-1}"'] & a \arrow[d, "s"] \\ x_n \arrow[r, "ki_n"'] \arrow[ur, dotted, "j_n"] & b \end{tikzcd}$$

By commutativity of all the upper triangles in these diagrams, the maps $\{j_n\}_{n \in \mathbb{N}}$ are compatible with the maps $\{f_n\}_{n \in \mathbb{N}}$ and induce by the universal property of the colimit a map $\ell : x \to a$. We have $s\ell = k$ by uniqueness in the universal property (applied to family of maps $\{sj_n\}_{n \in \mathbb{N}}$ and $\{ki_n\}_{n \in \mathbb{N}}$ respectively), and $\ell i_0 = h$ by construction (since $j_0 = h$).

□

Combining propositions (2.9) and (2.10), we get:

Proposition 2.11. *The class of cofibrations, respectively acyclic cofibrations, in a model category is stable under cobase change. Dually, the class of fibrations, respectively acyclic fibrations, is stable under base change.*

Proof. Apply proposition (2.10) to the class of maps $S = \mathbf{We}(C) \cap \mathbf{Fib}(C)$. Proposition (2.9) tells us that in this case $L = \mathbf{Cofib}(C)$. Thus the cobase change of a map of L (a cofibration) is again a map of L (a cofibration). Similarly, for the second statement we can choose $S = \mathbf{Fib}(C)$. The last two statements follow by duality. □

3 The homotopy category of a model category

3.1 Construction via localizations

In **Top**, a continuous map $f : X \to Y$ is a homotopy equivalence if and only if it admits an inverse up to homotopy, i.e. a continuous map $g : Y \to X$ such that $f \circ g$ is homotopic (in the usual topological sense) to id_Y and $g \circ f$ is homotopic to id_X. In some sense we would like to reproduce this situation for model categories and find inverses for weak equivalences. Categorical localizations are just about inverting a certain class of morphisms:

Definition 3.1 (Localization)**.** Let C be a category and W some class of morphisms in C. A *localization of C with respect to W* is the data of a category D and a functor $F : C \to D$ such that F sends morphisms in W to isomorphisms in D, and satisfies the following universal property: for any category D' and functor $G : C \to D'$ sending morphisms of W to isomorphisms, there exists a unique functor $G' : D \to D'$ such that $G'F = G$:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[dr, "G"'] & D \arrow[d, dashed, "\exists!\, G'"] \\ & D' \end{tikzcd}$$

In some sense, D is "minimal with the property that all morphisms of W are given an inverse".

Example 3.2. Let R be a ring, let $r \in R\backslash\{0\}$ and $S = \{1, r, r^2, \dots\}$. If R is a domain, $0 \notin S$ and the set $[S^{-1}]R := \{x/s \mid x \in R, s \in S\}$, modulo the equivalence relation $x/s \sim x'/s'$ if and only if $xs' = sx'$, is a subring of the fraction field of R. If M is an R-module, the set $[S^{-1}]M = \{(1/s)\cdot m \mid m \in M, s \in S\}$, with the equivalence relation $(1/s)\cdot m \sim (1/s')\cdot m'$ if and only if $s' \cdot m = s \cdot m'$, can be endowed with an $[S^{-1}]R$-module structure (given by $(x/s)\cdot((1/s')\cdot m) = (1/(ss'))\cdot(x \cdot m)$ for all $x \in R$, $s, s' \in S$ and $m \in M$). This yields a map between R-modules and $[S^{-1}]R$-modules. Actually, it defines a functor F between the corresponding categories Mod_R and $\mathrm{Mod}_{[S^{-1}]R}$ of R- and $[S^{-1}]R$-modules respectively. This functor turns out to be a localization of Mod_R with respect to the morphisms of the form $t_{M,r} : M \to M$, $m \mapsto r \cdot m$ for M an R-module. For example, in $\mathrm{Mod}_{[S^{-1}]R}$, $t_{[S^{-1}]M,1/r}$ is an inverse for $t_{[S^{-1}]M,r}$. We will not discuss this example further, but it connects localizations in the context of rings and modules ($[S^{-1}]R$ is called the localization of R with respect to S) and categorical localizations.

Using this notion we would like to define the homotopy category as:

Definition 3.3 (Homotopy category)**.** Let C be a model category. Then the *homotopy category* of C denoted by $\mathrm{Ho}(C)$ is the localization of C with respect to the class of weak equivalences $\mathbf{We}(C)$.

By definition of a localization, the homotopy category of C comes together with a functor $\gamma_C : C \to \mathrm{Ho}(C)$ sending weak equivalences to isomorphisms.

In order for this definition to be consistent, we need existence and uniqueness of the localization. First, the localization when it exists is unique up to isomorphism: indeed, consider C and W as in definition (3.1) and suppose that $F : C \to D$ and $G : C \to D'$ are two localizations of C with respect to W. Then by definition there exists functors $G' : D \to D'$ and $F' : D' \to D$ such that:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[dr, "G"] \arrow[ddr, "F"'] & D \arrow[d, "G'"] \\ & D' \arrow[d, "F'"] \\ & D \end{tikzcd}$$

is commutative. Thus, the functors id_D and $F' \circ G'$ both make the big triangle in the diagram above commute. By uniqueness in the universal property, $F'G' = \mathrm{id}_D$. Similarly $G'F' = \mathrm{id}_{D'}$. Then D and D' are isomorphic as categories, in a compatible way with the functors F and G.

On the other hand, we imagine that a localization must always exists, at least for small categories. Heuristically: we could construct the localization as a category D with the same objects as C, and morphisms in D are obtained by "formally adding composition inverses to morphisms in W and completing in a free way": for each morphism $w : a \to b$ in W, consider a formal composition inverse $w' : b \to a$. The morphisms in D will be made of finite sequences of composable morphisms in C or inverses for morphisms in W, modulo the relations $ww' \sim \mathrm{id}_b$ and $w'w \sim \mathrm{id}_a$ for any morphism $w : a \to b$ in W, and modulo identification of the sequences corresponding to composition in C: for example if $f \circ g = h$ in C then we set $wfg \sim wh$ in D. A map in D between objects a and b would be represented by a (non-unique) finite "zig-zag" with $w_i \in W$ or an identity and f_i any map in C for $i \leq n$ (with $n \in \mathbb{N}$) (imagine reversing the arrows w_i using their formal inverses w_i' in D):

$$\begin{tikzcd}[column sep=small] & c_1 \arrow[dl, "w_1"'] \arrow[dr, "f_1"] & & c_2 \arrow[dl, "w_2"'] \arrow[dr, "f_2"] & & & & c_n \arrow[dl, "w_n"'] \arrow[dr, "f_n"] & \\ a & & d_1 & & d_2 & \dots & d_{n-1} & & b \end{tikzcd}$$

Choose the functor F to be an inclusion, and the functor G' in the universal property above can be obtained because in some sense we built D in a minimal way : G' would send a morphism coming from C to its image by G, and a formal inverse for a morphism in W to the inverse of the image by G of this morphism. Rather than proving this rigourously and relying on abstract existence properties of localizations, we will construct the homotopy category in a more explicit way, which will also allow us to understand what "homotopic maps" means in the context of model categories. For more details about localizations, see the book "Calculus of fractions and homotopy theory" by Gabriel and Zisman (1967). In Chapter I, they make rigourous this notion of formally inverting morphisms, and build a "homotopic category" for simplicial sets in Chapter IV.

3.2 Construction via homotopy of maps

This subsection provides the advertised more explicit construction of the homotopy category. We first define the notions of left and right homotopy, and then study their relationship. For the rest of this subsection, let C be a model category.

3.2.1 The notion of cylinder object and left-homotopy

Two continuous maps $f, g : X \to Y$ between topological spaces are called *homotopic* if there exists a continuous map $H : X \times I \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. To generalize this definition to all model categories, we need an equivalent notion for the cylinder $X \times I$. Even though any model category admits finite products, there is no obvious equivalent for I. So we define:

Definition 3.4 (Cylinder object)**.** Let X be some object of C. A *cylinder object for* X is an object $X \wedge I$ of C with maps:

$$\begin{tikzcd} X \amalg X \arrow[r, "\iota"] \arrow[rr, bend right, "\mathrm{id}_X + \mathrm{id}_X"'] & X \wedge I \arrow[r, "p", "\sim"'] & X \end{tikzcd}$$

(p must be a weak equivalence and $p\iota$ must be equal to the map $\mathrm{id}_X + \mathrm{id}_X$ induced by the universal property of the coproduct). If we have the additional property that ι is a cofibration, $X \wedge I$ is called a *good cylinder object for* X and if also p is a fibration, $X \wedge I$ is a *very good cylinder object for X.*

Example 3.5. If we admit that the structure described in example (2.5) is a model structure on **Top**, then the usual cylinder $X \times I$ is a cylinder object for the space X: we can choose p to be the projection $p : X \times I \to X$ sending every (x,t) to x : this is a homotopy equivalence, and if we choose ι to be the inclusion $X \amalg X = (X \times \{0\}) \cup (X \times \{1\}) \subseteq X \times I$, we have $p \circ \iota = \mathrm{id}_X + \mathrm{id}_X$ as desired.

Given some object X, a cylinder object for X always exists in C. Indeed, since C admits coproducts, $X \amalg X$ exists. The diagram

$$X \amalg X \xrightarrow{\mathrm{id}_X + \mathrm{id}_X} X \;=\!=\!=\; X$$

shows that X is a cylinder object for itself (by MC0, id_X is a weak equivalence). We can do better than that: if we apply the factorization axiom MC5 to the map $\mathrm{id}_X + \mathrm{id}_X$, we get a very good cylinder object

$$X \amalg X \xhookrightarrow{\iota'(\mathrm{id}_X+\mathrm{id}_X)} X \wedge I := \omega'(\mathrm{id}_X + \mathrm{id}_X) \xrightarrow[\sim]{\pi'(\mathrm{id}_X+\mathrm{id}_X)} X$$

Now we have a notion of homotopy:

Definition 3.6 (Left-homotopic)**.** Consider two maps $f, g : X \to Y$ in C. We say that f is *left-homotopic* to g if there exists a ((very)good) cylinder object $X \amalg X \xrightarrow{\iota} X \wedge I \xrightarrow[\sim]{p} X$ for X and a map $H : X \wedge I \to Y$ (a *((very)good) left homotopy from f to g*) such that if $X \xrightarrow{i_0} X \amalg X \xleftarrow{i_1} X$ is the coproduct, then $H\iota i_0 = f$, and $H\iota i_1 = g$, equivalently $H \circ \iota = f + g$.

In this situation, we write $f \overset{\ell}{\sim} g$. We may loosely say that "f and g are left homotopic" because the relation $\overset{\ell}{\sim}$ is symmetric: keeping the notation of the definition, suppose that $f \overset{\ell}{\sim} g$ with homotopy H and cylinder object $X \wedge I$. Now, consider the same object $X \wedge I$ but with maps p and $\iota \circ (i_1 + i_0)$ this time ($i_1 + i_0$ "exchanges the two copies of X in the coproduct"). Then, H becomes a homotopy from g to f through this new cylinder object: indeed, $H \circ (\iota \circ (i_1 + i_0)) = (H \circ \iota) \circ (i_1 + i_0) = (f + g) \circ (i_1 + i_0) = g + f$ because $(f + g) \circ (i_1 + i_0) \circ i_0 = (f + g) \circ i_1 = g$ and $(f + g) \circ (i_1 + i_0) \circ i_1 = (f + g) \circ i_0 = f$ (we use the uniqueness in the universal property of the coproduct). The relation $\overset{\ell}{\sim}$ is also reflexive: for $f : X \to Y$, choose X as a cylinder object for itself, and $H = f$. Then $H \circ (\mathrm{id}_X + \mathrm{id}_X) = f + f$, we get $f \overset{\ell}{\sim} f$. This is basically the only interest of the cylinder object X for X: any homotopy H through this object is a homotopy between H and H.

However, left homotopy is not in general an equivalence relation. A key point when proving transitivity of homotopy between topological maps is that we can "stack" two cylinders $X \times I$ and obtain $X \times [0, 2] \approx X \times I$: this is like a concatenation of homotopies, and corresponds to the pushout of the diagram $X \times I \xleftarrow{\iota i_1} X \xrightarrow{\iota i_0} X \times I$. Let us try to reproduce the same proof here. Suppose that $f, g, h : X \to Y$ are maps in C with $f \overset{\ell}{\sim} g$ and $g \overset{\ell}{\sim} h$. Choose some homotopies H from f to g through a cylinder object $X \amalg X \xrightarrow{\iota} X \wedge I \xrightarrow[\sim]{p} X$ and H' from g to h through a cylinder object $X \amalg X \xrightarrow{\iota'} (X \wedge I)' \xrightarrow[\sim]{p'} X$. By axiom MC1, C has small colimits, so we may form the pushout of $X \wedge I \xleftarrow{\iota \circ i_1} X \xrightarrow{\iota' \circ i_0} (X \wedge I)'$. This induces maps (since $p'\iota' i_0 = p\iota i_1 = \mathrm{id}_X$ and $H\iota i_1 = H'\iota' i_0 = g$):

$$\begin{tikzcd}
X \arrow[r, "\iota\circ i_1"] \arrow[d, "\iota'\circ i_0"'] & X\wedge I \arrow[d, "j_0"] \arrow[ddr, bend left, "p"] & \\
(X\wedge I)' \arrow[r, "j_1"] \arrow[drr, bend right, "p'"'] & (X\wedge I)'' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! p''"] & \\
& & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\iota\circ i_1"] \arrow[d, "\iota'\circ i_0"'] & X\wedge I \arrow[d, "j_0"] \arrow[ddr, bend left, "H"] & \\
(X\wedge I)' \arrow[r, "j_1"] \arrow[drr, bend right, "H'"'] & (X\wedge I)'' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! H''"] & \\
& & Y
\end{tikzcd}$$

Consider the diagram $X \amalg X \xrightarrow{j_0\iota i_0 + j_1\iota' i_1} (X \wedge I)'' \xrightarrow{p''} X$. This diagram factorizes $\mathrm{id}_X + \mathrm{id}_X$: by construction, $p'' \circ j_0\iota i_0 = (p'' j_0) \circ \iota i_0 = p\iota i_0 = (\mathrm{id}_X + \mathrm{id}_X) i_0 = \mathrm{id}_X$ and similarly $p'' \circ j_1 \iota' i_1 = \mathrm{id}_X$. Moreover, $H'' \circ (j_0\iota i_0 + j_1\iota' i_1) = H'' j_0 \iota i_0 + H'' j_1 \iota' i_1 = H\iota i_0 + H'\iota' i_1 = (f + g) i_0 + (g + h) i_1 = f + h$.

Except for one detail, H'' is the desired homotopy between f and h: a priori p'' is not a weak equivalence, so $(X \wedge I)''$ might not be a cylinder object for X. Actually, we have the result:

Proposition 3.7 ($\overset{\ell}{\sim}$ as an equivalence relation)**.** *Given objects X, Y in C with X cofibrant, the left homotopy relation $\overset{\ell}{\sim}$ is an equivalence relation on the set $C(X, Y)$.*

Proof. We proved above that $\overset{\ell}{\sim}$ was reflexive and symmetric. We keep the notation of the reasoning above. To begin with, we would like to upgrade H and H' to good homotopies. We do the proof for H, and the result follows for H' in the exact same way. We apply the factorization axiom MC5 to the map ι: it may be factored as $q \circ j$ where j is a cofibration and q an acyclic fibration. We have the following situation:

$$\begin{tikzcd} X \amalg X \arrow[r, hook, "j"] \arrow[rr, bend left, "\iota"] \arrow[rrr, bend right, "\mathrm{id}_X + \mathrm{id}_X"'] & Z \arrow[r, two heads, "q", "\sim"'] & X \wedge I \arrow[r, two heads, "p", "\sim"'] & X \end{tikzcd}$$

Then Z is a good cylinder object for X with maps j and pq ($p \circ q$ is a weak equivalence by axiom MC0) and the homotopy $H : X \wedge I \to Y$ from f to g gives a homotopy $H \circ q$ between f and g through Z: $Hq \circ j = H\iota = f + g$ by hypothesis. This shows that, without loss of generality, we may assume that the cylinder objects $X \wedge I$ and $(X \wedge I)'$ are in fact good cylinder objects (i.e. ι and ι' are cofibrations), and that H and H' are good homotopies. Relying on the reasoning we did above, we only have to show that p'' is a weak equivalence. First, $\iota \circ i_1$ is a cofibration: by definition of a coproduct, the following diagram is a pushout:

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "i_0"] \\ X \arrow[r, "i_1"'] \arrow[ur, phantom, "\ulcorner"] & X \amalg X \end{tikzcd}$$

Because X is cofibrant, i_1 is the cobase change of a cofibration, i.e. it is a cofibration by proposition (2.11). By axiom MC0, the composition $\iota \circ i_1$ is a cofibration. We now apply the 2 of 3 rule to the maps ιi_1, p and $p \circ \iota i_1 = \mathrm{id}_X$: since p and id_X are weak equivalences, we get that ιi_1 is a weak equivalence too: it is an acyclic cofibration. Then j_1 as its cobase change is an acylic cofibration too, in particular it is a weak equivalence, and the 2 of 3 rule applied to p'', j_1 and $p'' j_1 = p'$ implies that p'' is a weak equivalence, since j_1 and p' are weak equivalences. Finally, $(X \wedge I)''$ defined above is a cylinder object for X and we are done. $\square$

Remark 3.8. While proving the proposition, we saw that any homotopy might be upgraded into a good homotopy. We also saw that if the source object X is cofibrant, the map $\iota i_1 : X \to X \wedge I$ is an acyclic cofibration. The same proof shows that ιi_0 is an acyclic cofibration too.

Definition 3.9 (Left homotopy classes)**.** For objects X and Y in C, $\pi^\ell(X, Y)$ denotes the set of equivalence classes of maps $C(X, Y)$ for the equivalence relation *generated* by $\overset{\ell}{\sim}$. The equivalence class of a map $f \in C(X, Y)$ is denoted by $[f]$ (or $[f]_\ell$ if we want to precise that we are considering left homotopy).

Remark 3.10. We saw that $\overset{\ell}{\sim}$ was not an equivalence relation in general, but it is transitive and reflexive. Therefore, for $f, g \in C(X, Y)$, we have $[f] = [g]$ in $\pi^\ell(X, Y)$ if and only if there exists a finite sequence of maps $f_1, \dots, f_k \in C(X, Y)$ such that $f \overset{\ell}{\sim} f_1 \overset{\ell}{\sim} f_2 \overset{\ell}{\sim} \dots \overset{\ell}{\sim} f_k \overset{\ell}{\sim} g$.

Here are two convenient properties of left homotopy, that seem pretty natural when thinking about topological spaces, with homotopy in the usual sense and homotopy equivalences as weak equivalences. Instead of proving them directly, we will prove their duals in the next subsection.

Lemma 3.11 (Bijection on left homotopy classes)**.** *Let X, Y and Z be objects in C. If X is cofibrant and $p : Y \overset{\sim}{\twoheadrightarrow} Z$ is an acyclic fibration, then p induces a bijection (by post-composition):*

$$\begin{aligned} p_* : \pi^\ell(X, Y) &\to \pi^\ell(X, Z) \\ [f] &\mapsto [pf] \end{aligned}$$

Lemma 3.12 (Composition by left homotopic maps)**.** *Let X, Y be objects in C and $f, g : X \to Y$ be maps. If Y is fibrant and $f \overset{\ell}{\sim} g$, then for any map $h : W \to X$, $fh \overset{\ell}{\sim} gh$.*

Remark 3.13. We will see (also by duality) that for any $h' : Y \to Z$, we have $h'f \overset{\ell}{\sim} h'g$ as soon as $f \overset{\ell}{\sim} g$, even without the assumption that Y is fibrant.

3.2.2 The notion of path object and right-homotopy

So far we have obtained a bijection by post-composition on homotopy equivalence classes, and some nice properties on "one side". We would like to have a similar bijection by pre-composition, and properties on the other side: the homotopy category will be constructed using homotopy classes of maps, and we would like to collect the maximal number of good properties. So we dualize the construction of left homotopy, following the ideas of subsection (2.2.1). For instance, if we think about a cylinder object $X \amalg X \xrightarrow{\iota} X \wedge I \xrightarrow[\sim]{p} X$, for some object X in C, when reversing the arrows, the coproduct $X \amalg X$ becomes the product $X \times X$, we get a map from some object X^I (the *path object*, replacing $X \wedge I$) to $X \times X$ and a weak equivalence $X \to X^I$, whose composition factorizes the dual of $\mathrm{id}_X + \mathrm{id}_X$, namely the map $(\mathrm{id}_X, \mathrm{id}_X) : X \to X \times X$. This leads to the definition:

Definition 3.14 (Path object)**.** Let X be some object of C. A *path object for* X is an object X^I of C with maps:

$$\begin{tikzcd} X \arrow[r, "\iota", "\sim"'] \arrow[rr, bend right, "(\mathrm{id}_X, \mathrm{id}_X)"'] & X^I \arrow[r, "p"] & X \times X \end{tikzcd}$$

If we have the additional property that p is a fibration, X^I is called a *good path object for* X and if also ι is a cofibration, X^I is a *very good path object for* X.

As with the cylinder objects, X is always a path object for X, and a very good path object for X can always be found using the factorization axiom.

Example 3.15. The notation X^I recalls the topological space of continuous functions from the interval I to some topological space X, endowed with the compact-open topology. If we think of weak equivalences as homotopy equivalences or weak homotopy equivalences, this space is a path object for X in **Top**. For K compact in I and U open in X, let $V(K,U) = \{f \in X^I \mid f(K) \subseteq U\}$. Then the set of all such $V(K,U)$ is a subbasis for the compact-open topology. Now, the map ι can be chosen as the map sending $x \in X$ to the constant map $\underline{x} : I \to X,\ t \mapsto x$, and p can be chosen as $(\mathrm{ev}_0, \mathrm{ev}_1) : X^I \to X \times X,\ f \mapsto (f(0), f(1))$, the evaluation at the two endpoints. The evaluation maps are continuous: if $U \subseteq X$ is open, then $ev_0^{-1}(U) = \{f : I \to X \mid f(0) \in U\} = V(\{0\}, U)$ is open in X^I, since $\{0\}$ is compact in I. For ev_1 the argument is the same. If $V(K,U)$ is an element of the subbasis above, $\iota^{-1}(V(K,U)) = \{x \in X \mid \underline{x}(K) \subseteq U\} = U$ is open in X, so ι is continuous. And ι is a homotopy equivalence (in particular, a weak homotopy equivalence) with homotopy inverse ev_0. Indeed, the composition $ev_0 \circ \iota$ is the identity, and $\iota \circ ev_0$ is homotopic to the identity via

$$\begin{aligned} H : X^I \times I &\longrightarrow X^I \\ (f,t) &\mapsto g : I \to X,\ s \mapsto \begin{cases} f(s) & s \leq t \\ f(t) & s \geq t \end{cases} \end{aligned}$$

because $H(f,0) = \iota(f(0)) = (\iota \circ ev_0)(f)$ and $H(f,1) = f$ for any $f \in X^I$. Also H is continuous: it suffices to check that the preimages of the sets in the subbasis given above are open. For $V(K,U)$ in the subbasis, we have

$$Z := H^{-1}(V(K,U)) = \{(f,t) \in X^I \times I \mid f(K \cap [0,t]) \subseteq U \text{ and } f(t) \in U \text{ if } K \cap [t,1] \neq \emptyset\}.$$

Let $(f,t) \in Z$. Assume first that $K \cap [t,1] = \emptyset$. Then, $t \in I \setminus K$, this is an open set in I, so there exists $\varepsilon > 0$ with $(t-\varepsilon, t+\varepsilon) \cap I \subseteq I \setminus K$. In this situation, $V(K \cap [0,t], U) \times (t-\varepsilon, t+\varepsilon)$ is an open neighbourhood of (f,t) in $X^I \times I$, contained in Z. Now assume that $K \cap [t,1] \neq \emptyset$. In particular, $(f,t) \in Z$ implies $f(t) \in U$, and since U is open and f continuous, there exists $\varepsilon > 0$ with $f((t-2\varepsilon, t+2\varepsilon) \cap I) \subseteq U$. Then, $V((K \cap [0, t+\varepsilon]) \cup [t-\varepsilon, t+\varepsilon], U) \times (t-\varepsilon, t+\varepsilon)$ is an open neighbourhood of (f,t) in $X^I \times I$, contained in Z. By arbitrarity of (f,t), Z is open and we are done.

We also get the dual notation of homotopy:

Definition 3.16 (Right-homotopic)**.** Consider two maps $f, g : X \to Y$ in C. We say that f is *right-homotopic* to g if there exists a ((very)good) path object $Y \xrightarrow[\sim]{\iota} Y^I \xrightarrow{p} Y \times Y$ for Y and a map $H : X \to Y^I$ (a *((very) good) right homotopy between f and g*) such that $p \circ H = (f, g)$.

Again $f \overset{r}{\sim} g$ denotes the fact that f and g are right homotopic, and for two objects X and Y, let $\pi^r(X, Y)$ be the set of equivalence classes for the equivalence relation *generated* by $\overset{r}{\sim}$ on $C(X, Y)$.

For left homotopy the properties of the source object were crucial in determining whether $\overset{\ell}{\sim}$ what an equivalence relation or not (and we were considering a cylinder object for the *source* X). By duality, here the *target* object Y will matter more, and we have the property:

Proposition 3.17 ($\overset{r}{\sim}$ as an equivalence relation)**.** *Given objects X, Y in C with Y fibrant, the right homotopy relation $\overset{r}{\sim}$ is an equivalence relation on $C(X, Y)$.*

The duals of lemmas (3.11) and (3.12) are:

Lemma 3.18 (Bijection on right homotopy classes)**.** *Let X, Y and Z be objects in a model category C. If Z is fibrant and $\iota : X \overset{\sim}{\hookrightarrow} Y$ is an acyclic cofibration then ι induces a bijection (by pre-composition):*

$$\begin{aligned} \iota_* : \pi^r(Y, Z) &\to \pi^r(X, Z) \\ [f] &\mapsto [f\iota] \end{aligned}$$

Proof. First, notice that since Z is fibrant, by the preceding proposition (3.17), $\overset{r}{\sim}$ is an equivalence relation on $C(Y, Z)$ and $C(X, Z)$. Thus, for $f, g \in C(Y, Z)$, $[f] = [g]$ in $\pi^r(Y, Z)$ if and only if f and g are right homotopic (directly, not with a sequence $f \overset{r}{\sim} f' \overset{r}{\sim} \dots \overset{r}{\sim} g$).

The map ι_* is well defined: if $[f] = [g]$ in $\pi^r(Y, Z)$, there exists a path object $Z \xrightarrow[\sim]{j} Z^I \xrightarrow{q} Z \times Z$ for Z and a homotopy $H : Y \to Z^I$ with $qH = (f, g)$. Then $f\iota \overset{r}{\sim} g\iota$: indeed, $H\iota : X \to Z^I$ is a right homotopy between $f\iota$ and $g\iota$ through the path object Z^I because $qH\iota = (f, g)\iota = (f\iota, g\iota)$.

Moreover ι_* is injective: consider $[f], [g] \in \pi^r(Y, Z)$ with $[f\iota] = [g\iota]$, i.e. $f\iota \overset{r}{\sim} g\iota$ with some good (dual of remark (3.8)) homotopy H through a good path object $Z \xrightarrow[\sim]{j} Z^I \overset{q}{\twoheadrightarrow} Z \times Z$ for Z. We want to show that $f \overset{r}{\sim} g$. By axiom MC4, a lift exists in the diagram:

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, hook, "\iota"', "\wr"] & Z^I \arrow[d, two heads, "q"] \\ Y \arrow[r, "(f,g)"'] \arrow[ur, dashed, "\exists\ h"] & Z \times Z \end{tikzcd}$$

so $qh = (f, g)$, i.e. h is a right homotopy between f and g through the path object Z^I, hence $[f] = [g]$.

For surjectivity, let $g : X \to Z$. We would like to find $f : Y \to Z$ such that $\iota_*([f]) = [g]$, i.e. $f\iota \overset{r}{\sim} g$. Since Z is fibrant and ι is an acyclic cofibration, a lift exists in the diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, hook, "\iota"', "\wr"] & Z \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed, "f"] & * \end{tikzcd}$$

and actually $f\iota = g$, in particular by reflexivity of $\overset{r}{\sim}$, $f\iota \overset{r}{\sim} g$ as desired. Hence ι_* is a bijection. ☐

Remark 3.19. We saw in the proof that for any maps $f, g : X \to Y$ with $f \overset{r}{\sim} g$, and any map $h : W \to X$, we have $fh \overset{r}{\sim} gh$.

Under an additional assumption there is a similar statement for post-composition by h:

Lemma 3.20 (Composition by right homotopic maps)**.** *Let X, Y be objects in C and $f, g : X \to Y$ be maps. If X is cofibrant and $f \overset{r}{\sim} g$, then for any map $h : Y \to Z$, $hf \overset{r}{\sim} hg$.*

Proof. Suppose $f \overset{r}{\sim} g$. There exists a good (dual of remark (3.8)) path object $Y \xrightarrow[\sim]{\iota} Y^I \overset{p}{\twoheadrightarrow} Y \times Y$ for Y and a good homotopy $H : X \to Y^I$ between f and g. Let $Z \xrightarrow[\sim]{\iota'} Z^I \overset{p'}{\twoheadrightarrow} Z \times Z$ be a very good path object for Z (found by factorization of $(\mathrm{id}_Z, \mathrm{id}_Z)$ by axiom MC5). We would like to find a homotopy $H' : X \to Z^I$ between hf and hg. Call π_0 and π_1 the natural maps $Y \times Y \to Y$ in the product. We would like to set $H' = \ell H$ where ℓ is a lift in :

$$\begin{tikzcd} Y \arrow[r, "\iota' h"] \arrow[d, "\iota"', "\wr"] & Z^I \arrow[d, two heads, "p'"] \\ Y^I \arrow[r, "(h\pi_0 p, h\pi_1 p)"'] \arrow[ur, dashed, "\ell"] & Z \times Z \end{tikzcd}$$

We would have $p'H' = p'\ell H = (h\pi_0 p, h\pi_1 p) \circ H = (h\pi_0 pH, h\pi_1 pH) = (h\pi_0(f,g), h\pi_1(f,g)) = (hf, hg)$ and would be done. However, we cannot always find a lift in the diagram above, except if ι is an acyclic cofibration. This is where the assumption that X is cofibrant comes in a handy: factor the map ι by axiom (MC5) as $Y \overset{j}{\underset{\sim}{\hookrightarrow}} Y^{I'} \overset{q}{\twoheadrightarrow} Y^I$. Since ι and j are weak equivalences, the 2 of 3 rule implies that q is a weak equivalence too. Also $Y \overset{j}{\underset{\sim}{\hookrightarrow}} Y^{I'} \overset{pq}{\twoheadrightarrow} Y \times Y$ is now a very good cylinder object for Y, and we have a very good homotopy $H'' : X \to Y^{I'}$ between f and g given by a lift in the diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & Y^{I'} \arrow[d, two heads, "q", "\wr"'] \\ X \arrow[r, "H"'] \arrow[ur, dashed, "\exists\ H''"] & Y^I \end{tikzcd}$$

We use crucially here that X is cofibrant. In this case we have $pqH'' = pH = (f, g)$ by construction. That is, we may assume that in the first part of the proof, Y^I was actually a very good path object for Y, in particular that ι was an acyclic cofibration, so that the desired lift exists, and this concludes the proof. $\square$

We have two different notions of homotopy, both with interesting properties. We would now like to gather this properties and unite the two notions into a single one.

3.2.3 Comparing the two types of homotopy

We discuss conditions under which left and right homotopy coincide. We saw in the two previous subsections that left and right homotopy have good properties when the source or the target objects, are cofibrant or fibrant respectively. Then, the assumptions of the following proposition are no surprise:

Proposition 3.21 (Relationship between $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$)**.** *Let $f, g : X \to Y$ be two maps in C. If X is cofibrant and $f \overset{\ell}{\sim} g$, then $f \overset{r}{\sim} g$, and dually if Y is fibrant and $f \overset{r}{\sim} g$, then $f \overset{\ell}{\sim} g$. If X is cofibrant and Y is fibrant, left and right homotopy coincide and define an equivalence relation $\sim$ on $C(X, Y)$.*

Proof. The second claim follows from the first one by duality, so we shall only prove the first one. Suppose that $f \overset{\ell}{\sim} g$. Since X is cofibrant, by proposition (3.7), they are really right homotopic, and by remark (3.8), there exists a good cylinder object $X \amalg X \overset{\iota_0 + \iota_1}{\hookrightarrow} X \wedge I \xrightarrow[\sim]{p} X$ with a good homotopy $H : X \wedge I \to Y$ between f and g. In particular $H\iota_0 = f$ and $H\iota_1 = g$. Let Y^I be some good path object for Y: $Y \xrightarrow[\sim]{\iota'} Y^I \overset{p'}{\twoheadrightarrow} Y \times Y$. We are looking for a homotopy $H' : X \to Y^I$ with $p'H' = (f, g)$. Since X is cofibrant, by remark (3.8), ι_0 is an acyclic cofibration. Then, a lift exists in the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\iota_0"', "\wr"] & Y \arrow[r, "\iota'"] & Y^I \arrow[d, two heads, "p'"] \\ X \wedge I \arrow[rr, "(fp, H)"'] \arrow[urr, dashed, "\exists h"] & & Y \times Y \end{tikzcd}$$

Consider $H' := h\iota_1$. Then $p'H' = p'h\iota_1 = (fp, H) \circ \iota_1 = (fp\iota_1, H\iota_1) = (f, g)$ (by definition of a cylinder object, $p\iota_1 = \mathrm{id}_X$). In conclusion $f \overset{r}{\sim} g$.

For the second part of the proposition: since X is cofibrant and Y is fibrant, by propositions (3.7) and (3.17), left and right homotopy are equivalence relations on $C(X, Y)$. By the first part of the statement, for $f, g \in C(X, Y)$ we have that $f \overset{\ell}{\sim} g$ if and only if $f \overset{r}{\sim} g$. Then right and left homotopy coincide, and we may denote by $\sim$ the equivalence relation corresponding to both $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$. $\square$

Remark 3.22. Here the good path object Y^I was arbitrary. Hence, if we fix ourselves a good path object Y^I, in the situation where X is cofibrant and Y is fibrant:

$$f \overset{\ell}{\sim} g \iff f \sim g \iff f \overset{r}{\sim} g \text{ through } Y^I \iff f \overset{r}{\sim} g,$$

and dually for fixed good cylinder objects.

An important step in the direction of remark (2.6) (we want weak equivalences to correspond to invertible maps in the homotopy category) is:

Proposition 3.23 (Weak equivalences and homotopy inverses I)**.** *Consider two bifibrant objects X and Y in C. Then a map $f \in C(X, Y)$ is a weak equivalence if and only if there exists $g \in C(Y, X)$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.*

Proof.
- By the previous proposition (3.21), since X and Y are bifibrant, left and right homotopy coincide and define an equivalence relation $\sim$ on $C(X, Y)$ as well as on $C(Y, X)$. We will use remarks (3.13) and (3.19) a lot in the rest of the proof without explicitly mentioning it.

- For the first implication, assume that $f \in C(X, Y)$ is a weak equivalence. By the factorization axiom MC5, we may write f as a composition $X \overset{\iota}{\underset{\sim}{\hookrightarrow}} Z \overset{p}{\twoheadrightarrow} Y$. Since $p\iota = f$ and ι are weak equivalences, by the 2 of 3 rule p is a weak equivalence too. Note that Z is also bifibrant: the unique map $\emptyset \to Z$ is obtained as the composition of the two cofibrations $\emptyset \hookrightarrow X$ and ι, so it is a cofibration, and the unique map $Z \to *$ is obtained as the composition of the two fibrations p and $Y \twoheadrightarrow *$, so it is a fibration (by MC0). We will use the properties of the maps p and ι to find a left/right inverse for them, and the composition of these inverses will become a homotopy inverse for f. To do so, we find lifts in the following diagrams:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, hook, "\iota"', "\wr"] & X \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed, "\exists\,\iota'"] & * \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & Z \arrow[d, two heads, "p", "\wr"'] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\exists\,p'"] & Y \end{tikzcd}$$

Actually, ι' is a homotopy inverse for ι: on the one hand, $\iota' \circ \iota = \mathrm{id}_X$. On the other hand, ι induces a bijection $\iota_* : \pi^r(Z, Z) \to \pi^r(X, Z)$ by precomposition, using proposition (3.18) (ι is an acyclic cofibration and Z is fibrant). We want to show that $[\mathrm{id}_Z] = [\iota \circ \iota']$. We have $\iota_*([\mathrm{id}_Z]) = [\iota] = [\iota \circ (\iota' \circ \iota)] = [(\iota \circ \iota') \circ \iota] = \iota_*([\iota \circ \iota'])$. Injectivity of ι_* yields the result.

Similarly, p' is a homotopy inverse for p: on the one hand, $p \circ p' = \mathrm{id}_Y$ and on the other hand, p induces a bijection $p_* : \pi^\ell(Z, Z) \to \pi^\ell(Z, Y)$ (by postcomposition, by proposition (3.11) using the fact that p is an acyclic fibration and Z is cofibrant), and $p_*([p'p]) = [pp'p] = [p] = p_*([\mathrm{id}_Z])$ hence by injectivity pf p_*, we have $[p' \circ p] = [\mathrm{id}_Z]$ as desired.

Now $\iota' p'$ is a homotopy inverse for f because

$$\begin{aligned} & [f \circ \iota' p'] = [p\iota\iota' p'] = p_*([\iota\iota' p']) = p_*([\mathrm{id}_Z p']) = p_*([p']) = [pp'] = [\mathrm{id}_Y] \\ \text{and} \quad & [\iota' p' \circ f] = [\iota' p' p\iota] = \iota_*([\iota' p' p]) = \iota_*([\iota' \mathrm{id}_Z]) = \iota_*([\iota']) = [\iota'\iota] = [\mathrm{id}_X] \end{aligned}$$

- For the other implication, suppose that $g : Y \to X$ is a homotopy inverse for f. By the factorization axiom MC5, we may write f as $X \overset{\iota}{\underset{\sim}{\hookrightarrow}} Z \overset{p}{\twoheadrightarrow} Y$, where for the same reasons as above Z is bifibrant. To show that f is a weak equivalence, it suffices to show that p is a weak equivalence (f will be a composition of weak equivalences). Our strategy is to express p as a retract of a weak equivalence, so that we can conclude by the retract axiom MC3.

Consider a good homotopy $H : Y \wedge I \to Y$ between fg and id_Y for some good cylinder object $Y \amalg Y \xhookrightarrow{q_0+q_1} Y \wedge I \xrightarrow[\sim]{r} Y$. Since Y is cofibrant, q_0 is an acyclic cofibration (by remark (3.8)). Therefore a lift exists in the following diagram (since $p\iota g = fg = Hq_0$):

$$\begin{tikzcd} Y \arrow[r, "\iota g"] \arrow[d, hook, "q_0"', "\wr"] & Z \arrow[d, two heads, "p"] \\ Y \wedge I \arrow[r, "H"'] \arrow[ur, dashed, "\exists\, H'"] & Y \end{tikzcd}$$

By the preceding part of the proof, since ι is a weak equivalence between bifibrant objects, it has a homotopy inverse $\iota' : Z \to X$. Now the map $\theta := H'q_1p$ happens to be a weak equivalence and p is a retract of θ, so we can conclude as explained above. Indeed:

- H' is a (left) homotopy between ιg and $H'q_1 : H'(q_0 + q_1) = H'q_0 + H'q_1 = \iota g + H'q_1$. We have: $\theta = H'q_1p \sim \iota gp \sim \iota gf\iota' \sim \iota\iota' \sim \mathrm{id}_Z$ where we used the fact that $p \sim f\iota'$ since their image by the bijection ι_* (see proposition (3.18)) is equal ($\iota_*([p]) = [p\iota] = [f]$ and $\iota^*([f\iota']) = [f\iota'\iota] = [f]$ because $\iota'\iota \sim \mathrm{id}_X$).
- $\theta \sim \mathrm{id}_Z$ implies that θ is a weak equivalence: there is a good cylinder object $Z \wedge I$ for Z (with a map $Z \xhookrightarrow{j_0+j_1} Z \wedge I$) and a good homotopy $H'' : Z \wedge I \to Z$ such that $H''j_0 = \theta$ and $H''j_1 = \mathrm{id}_Z$. Since Z is bifibrant, by the remark (3.8), j_0 and j_1 are acyclic cofibrations. Therefore, since id_Z is a weak equivalence, by the 2 of 3 rule for $\mathrm{id}_Z = H''j_1$, H'' and j_1 we get that H'' is a weak equivalence. Then $\theta = H''j_0$ is a weak equivalence by MC0.
- The following diagram expresses p as a retract of θ:

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "\theta"', "\wr"] & Z \arrow[d, two heads, "p"] \\ Z \arrow[r, two heads, shift right, "p"'] & Y \arrow[l, hook', shift right, "H'q_1"'] \end{tikzcd}$$

  because $pH'q_1 = Hq_1 = \mathrm{id}_Y$ and $p\theta = pH'q_1p = \mathrm{id}_Y p = p$.

$\square$

Example 3.24. If we apply this result to the Quillen model structure on **Top**, described in section (5), we obtain *Whitehead's theorem*, which states that every weak homotopy equivalence (a weak equivalence for this model structure) between CW-complexes (which happen to be bifibrant objects) is a homotopy equivalence (i.e. has a homotopy inverse, in the usual topological sense): in this situation usual homotopy coincides with homotopy in the sense of the model structure.

3.2.4 (Co)fibrant replacements

In order for the desirable situation of the previous subsection to happen, we need to replace our objects with bifibrant versions of themselves, so that any map induces a map between the replacements of the source and target object, preferably with good properties such as uniqueness or preservation of weak equivalences. This is what we aim to do in this subsection. This description recalls the notion of a functor, but for this we need a target category. So we define:

Definition 3.25 (Categories of (co)fibrant objects)**.** Let:

(i) πC_c the category with objects all cofibrant objects of C and morphisms $\pi C_c(X, Y) = \pi^r(X, Y)$ for any cofibrant objects X, Y in C.

(ii) πC_f the category with objects all fibrant objects of C and morphisms $\pi C_f(X, Y) = \pi^\ell(X, Y)$ for any fibrant objects X, Y in C.

(iii) πC_{cf} the category with objects all bifibrant objects of C and morphisms $\pi C_{cf}(X, Y) = \pi(X, Y)$ (left and right homotopy coincide in this situation) for any bifibrant objects X, Y in C.

with composition given by composition of representatives of the equivalence classes.

In (iii) left and right homotopy coincide by proposition (3.21) because all objects are bifibrant. We have to check that composition is well defined, for example we check this for point (i) and the other ones follow in a similar way. Consider some objects X, Y, Z in πC_c. We have to check that the map $\pi^r(X,Y) \times \pi^r(Y,Z) \to \pi^r(X,Z)$ given by $([f],[g]) \mapsto [gf]$ is well-defined, i.e. that if $[f]_r = [f']_r$ and $[g]_r = [g']_r$ for some maps $f, f' \in C(X,Y)$ and $g, g' \in C(Y,Z)$, then $[gf]_r = [g'f']_r$. A delicate point here is that $\overset{r}{\sim}$ might not be an equivalence relation on $C(X,Y)$ or $C(Y,Z)$ (it might miss transitivity). So $[f]_r = [f']_r$ and $[g]_r = [g']_r$ means that there exists $n, m \in \mathbb{N}^*$ and maps $f_1, \ldots, f_{n-1} \in C(X,Y)$, $g_1, \ldots, g_{m-1} \in C(Y,Z)$ such that $f_i \overset{r}{\sim} f_{i+1}$ for all $i \leq n$ and $g_i \overset{r}{\sim} g_{i+1}$ for all $i \leq m$ if we set $f_0 = f$, $f_n = f'$, $g_0 = g$ and $g_m = g'$. This gives a sequence $\{g_i \circ f_i\}_{i=0}^N$ where $N = \max\{n, m\}$ (we set $g_k = g'$ for all $k \geq m$ and $f_k = f'$ for all $k \geq n$) with first term gf and last term $g'f'$, so if we show $[g_i f_i]_r = [g_{i+1} f_{i+1}]_r$ for all $i \leq N-1$, we are done. Let $i \leq N-1$. This time, we reduced to a case where g_i is really right homotopic to g_{i+1} and similarly for f. By lemma (3.20), $f_i \overset{r}{\sim} f_{i+1}$ implies $g_i f_i \overset{r}{\sim} g_i f_{i+1}$ (all objects are cofibrant). Now by remark (3.19), the fact that $g_i \overset{r}{\sim} g_{i+1}$ implies that $g_i f_{i+1} \overset{r}{\sim} g_{i+1} f_{i+1}$. Therefore we get, by transitivity of the *generated* equivalence relation, that $[g_i f_i]_r = [g_{i+1} f_{i+1}]_r$ as desired.

We can now define the functors we need:

Proposition 3.26 (Cofibrant replacement)**.** *There exists a functor $Q : C \to \pi C_c$ sending each object X to a cofibrant object QX with a map $p_X : QX \overset{\sim}{\twoheadrightarrow} X$ (if X is already cofibrant we set $QX = X$), and each map $f : X \to Y$ to the equivalence class $Qf := [\tilde{f}] \in \pi C_c(QX, QY) = \pi^r(QX, QY)$, for a representative $\tilde{f} \in C(QX, QY)$ such that:*

$$\begin{tikzcd}
QX \arrow[r, "\tilde{f}"] \arrow[d, two heads, "p_X"', "\wr"] & QY \arrow[d, two heads, "p_Y", "\wr"'] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

And:

(i) in particular, the functor is well-defined and any two maps $\tilde{f}, \hat{f} : QX \to QY$ making the diagram above commute are left and right homotopic.

(ii) $\tilde{f}$ is a weak equivalence if and only if f is itself a weak equivalence.

(iii) if Y is fibrant and $f \overset{\ell}{\sim} g$ then $Qf = Qg$.

Proof. We will prove the proposition in the dual case. □

The dual version of proposition (3.26) is:

Proposition 3.27 (Fibrant replacement)**.** *There exists a functor $R : C \to \pi C_f$ sending each object X to a fibrant object RX with a map $\iota_X : X \overset{\sim}{\hookrightarrow} RX$ (if X is already fibrant we set $RX = X$), and each map $f : X \to Y$ to the equivalence class $Rf := [\overline{f}] \in \pi C_f(RX, RY) = \pi^\ell(RX, RY)$, for a representative $\overline{f} \in C(RX, RY)$ such that:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "\iota_X"', "\wr"] & Y \arrow[d, hook, "\iota_Y", "\wr"'] \\
RX \arrow[r, "\overline{f}"'] & RY
\end{tikzcd}$$

and:

(i) in particular, the functor is well-defined and any two maps $\overline{f}, \hat{f} : RX \to RY$ making the diagram above commute are left and right homotopic.

(ii) $\overline{f}$ is a weak equivalence if and only if f is itself a weak equivalence.

(iii) if X is cofibrant and $f \overset{r}{\sim} g$ then $Rf = Rg$.

Proof. For any object X in C, let $u_X : X \to *$ be the unique map from X to the terminal object of C. Axiom MC5 provides us with a choice of factorizations $X \overset{\iota(u_X)}{\underset{\sim}{\hookrightarrow}} \omega(u_X) \twoheadrightarrow *$ for all objects X in C. If X is not fibrant, set $RX = \omega(u_X)$ and $\iota_X = \iota(u_X)$. If X is already fibrant, we choose $RX = X$ and $\iota_X = \mathrm{id}_X$ (by MC0 it is an acyclic fibration).

For any $f : X \to Y$ in C, a map $\overline{f}$ as in the statement always exist: a lift exists in the following diagram:

$$\begin{tikzcd} X \arrow[r, "\iota_Y f"] \arrow[d, hook, "\iota_X"', "\wr"] & RY \arrow[d, two heads] \\ RX \arrow[r, two heads] \arrow[ur, dashed, "\exists\, \overline{f}"] & * \end{tikzcd}$$

and the map $\overline{f}$ we obtain satisfies by construction $\overline{f}\iota_X = \iota_Y f$.

The functor R is well defined: suppose that $\overline{f}, \hat{f} : RX \to RY$ both make the diagram of the statement commute. Then, their images under the bijection $(\iota_X)_* : \pi^r(RX, RY) \to \pi^r(X, RY)$ induced by the acyclic cofibration ι_X (since RY is fibrant, we can apply lemma (3.18)) are the same: $\overline{f}\iota_X = \iota_Y f = \hat{f}\iota_X$ hence $(\iota_X)_*([\overline{f}]_r) = [\overline{f}\iota_X]_r = [\hat{f}\iota_X]_r = (\iota_X)_*([\tilde{f}]_r)$. Thus $[\hat{f}]_r = [\overline{f}]_r$, and since RY is fibrant, $\overset{r}{\sim}$ is an equivalence relation on $C(RX, RY)$, implying that $\hat{f} \overset{r}{\sim} \overline{f}$. By proposition (3.21), we also get $\hat{f} \overset{\ell}{\sim} \overline{f}$, hence $[\hat{f}]_\ell = [\overline{f}]_\ell$ and R is well-defined.

We check functoriality: for two maps $f : X \to Y$, $g : Y \to Z$, if $\overline{f}$ and $\overline{g}$ are representatives for Rf and Rg then $\overline{g}\,\overline{f}$ makes the diagram in the statement for $g \circ f$ commutative. By the uniqueness up to homotopy in the last paragraph, $Rg \circ Rf = [\overline{g}]_\ell \circ [\overline{f}]_\ell = [\overline{g} \circ \overline{f}]_\ell = R(g \circ f)$ by definition of the composition in πC_f.

For (ii): by MC2, $f \in \mathbf{We}(C) \iff \iota_Y f \in \mathbf{We}(C) \iff \overline{f}\iota_X \in \mathbf{We}(C) \iff \overline{f} \in \mathbf{We}(C)$.

For (iii): by the same reasoning as above, showing that $[\overline{f}\iota_X]_r = [\overline{g}\iota_X]_r$ suffices to show that $Rf = [\overline{f}]_\ell = [\overline{g}]_\ell = Rg$. The equality we have to show rewrites $[\iota_Y f]_r = [\iota_Y g]_r$. But $f \overset{r}{\sim} g$ implies $\iota_Y f \overset{r}{\sim} \iota_Y g$ by lemma (3.20) since X is cofibrant by assumption. $\square$

Later on, when working with some map f, if we write $\tilde{f}$ or $\overline{f}$, we mean that we choose maps as in the statements of these propositions.

3.2.5 The construction

Now we are ready to construct the homotopy category. In order to obtain bifibrant objects only, we would like to apply successively functors Q and R. We need R to induce a functor from πC_c to πC_{cf} (which we also denote by R, by a small abuse of notation) so that we can compose it with Q. Two things need to be checked: one, that R sends cofibrant objects to cofibrant objects, and two, that R sends right homotopic maps between cofibrant objects to homotopic maps between their fibrant replacement. The second statement follows from property (iii) in proposition (3.27). For the first one, let X be a cofibrant object in C. If X is already fibrant, then $RX = X$ is cofibrant. Otherwise, we have constructed RX by factorizing the map $X \to *$, so we have the situation: $\emptyset \hookrightarrow X \overset{\iota_X}{\underset{\sim}{\hookrightarrow}} RX \twoheadrightarrow *$. Because $\emptyset$ is initial, the unique map $\emptyset \to RX$ is given by the composition of ι_X and $\emptyset \hookrightarrow X$, and is a cofibration as the composition of two cofibrations (axiom MC0). Hence RX is still cofibrant.

We are now ready to state the main definition:

Definition 3.28 (Homotopy category)**.** The *homotopy category* $\mathrm{Ho}(C)$ of C is the category with objects $\mathrm{Ob}(C)$, and for any objects X and Y in C, $\mathrm{Ho}(C)(X,Y) = \pi C_{cf}(RQX, RQY) = \pi(RQX, RQY)$. The functor $\gamma_C : C \to \mathrm{Ho}(C)$ is defined as the identity on objects and the composition RQ on arrows.

If X and Y are bifibrant objects, we have $RQX = X$ and $RQY = Y$, hence $\mathrm{Ho}(C)(X,Y)$ is exactly the set of equivalence classes for homotopy of maps from X to Y, hence the idea of a homotopy category. The functoriality of γ_C comes from the fact that it is given on maps by the composition of the two functors R and Q. An important step in the direction of remark (2.6) and on the continuation of proposition (3.23) is:

Proposition 3.29 (Weak equivalences and homotopy inverses II)**.** *If $f : X \to Y$ is a map in C, $\gamma_C(f)$ is an isomorphism in $\mathrm{Ho}(C)$ if and only if f is a weak equivalence. Moreover, the morphisms of $\mathrm{Ho}(C)$ are generated under composition by morphisms in the image of γ_C and inverses of images by γ_C of weak equivalences.*

Proof. Suppose $f : X \to Y$ is a weak equivalence. Then Qf admits a weak equivalence $\tilde{f} : QX \to QY$ as a representative by proposition (3.26), and RQf also admits a weak equivalence $\overline{\tilde{f}} : RQX \to RQY$ as a representative, by proposition (3.27). Since RQX and RQY are bifibrant, proposition (3.23) implies that $\overline{\tilde{f}}$ admits a homotopy inverse $g : RQY \to RQX$. Then, an inverse for $\gamma_C(f) = RQf = \left[\overline{\tilde{f}}\right]$ is given by $[g] \in \pi(RQY, RQX) = \mathrm{Ho}(C)(X, Y)$. Conversely, if $\gamma_C(f) = RQf = \left[\overline{\tilde{f}}\right]$ is invertible in $\mathrm{Ho}(C)$ with inverse $[g]$, it means that the representative $\overline{\tilde{f}}$ admits g as a homotopy inverse and again by proposition (3.23), $\overline{\tilde{f}}$ is a weak equivalence, which implies by applying in turns propositions (3.27) and (3.26) that f and f are weak equivalences.

For the second part of the statement, let $\theta \in \mathrm{Ho}(C)(X, Y)$ and consider the composition in $\mathrm{Ho}(C)$:

$$\begin{tikzcd}
RQX \arrow[r, "\gamma_C(\iota_{QX})^{-1}", "\cong"'] & QX \arrow[r, "\gamma_C(p_X)", "\cong"'] & X \arrow[d, "\theta"] \\
RQY & QY \arrow[l, "\cong", "\gamma_C(\iota_{QY})"'] & Y \arrow[l, "\cong", "\gamma_C(p_Y)^{-1}"']
\end{tikzcd}$$

Since $\mathrm{Ho}(C)(RQX, RQY) = \pi(RQX, RQY)$, there exists $f : RQX \to RQY$ in C such that $[f]$ is equal to the composition above, and $\gamma_C(f) = \left[\overline{\tilde{f}}\right] = [f]$ (RQX and RQY are already bifibrant) so in the end $\theta = \gamma_C(p_Y)\gamma_C(\iota_{QY})^{-1}\gamma_C(f)\gamma_C(\iota_{QX})\gamma_C(p_X)^{-1}$. By arbitrarity of θ this shows the claim. □

If the source and target objects A and B have additional properties, there is a nicer description of $\mathrm{Ho}(C)(A, B)$:

Lemma 3.30 (Nicer description of maps in $\mathrm{Ho}(C)$)**.** *Suppose that A is cofibrant and B is fibrant. Then $\mathrm{Ho}(C)(A, B)$ is in bijection (induced by the maps p_B and ι_A) with $\pi(A, B)$ (equivalence classes of maps in C).*

Proof. First, by proposition (3.21), left and right homotopy coincide on $C(A, B)$ and define an equivalence relation. So $\pi(A, B) = \pi^r(A, B) = \pi^\ell(A, B)$.

We have $\mathrm{Ho}(C)(A, B) = \pi(RQA, RQB) = \pi(RA, QB)$ because A is already cofibrant and QB is already fibrant (in the same way that R preserves cofibrant objects, Q preserves fibrant objects). Since ι_A is an acyclic cofibration and QB is fibrant, ι_A induces by lemma (3.18) a bijection $(\iota_A)_* : \pi^r(RA, QB) \to \pi^r(A, QB)$. Since A is cofibrant and QB is fibrant, by proposition (3.21), $\pi^r(A, QB) = \pi^\ell(A, QB)$. Since p_B is an acyclic fibration and A is cofibrant, p_B induces a bijection $(p_B)_* : \pi^\ell(A, QB) \to \pi^\ell(A, B)$ (by lemma (3.11)). Finally we have the bijection:

$$\mathrm{Ho}(C)(A, B) = \pi(RA, QB) = \pi^r(RA, QB) \xrightarrow{(\iota_A)_*} \pi^r(A, QB) = \pi^\ell(A, QB) \xrightarrow{(p_B)_*} \pi^\ell(A, B) = \pi(A, B)$$

□

3.3 Equivalence of the two constructions

As claimed at the end of subsection 3.1:

Theorem 3.31 (Equivalence of the two constructions of $\mathrm{Ho}(C)$)**.** *The homotopy category $\mathrm{Ho}(C)$ of a model category together with the functor $\gamma_C : C \to \mathrm{Ho}(C)$ (cf definition (3.28)) is a localization of C with respect to the class $\mathbf{We}(C)$ of weak equivalences.*

Proof. By proposition (3.29), the functor $\gamma_C : C \to \mathrm{Ho}(C)$ sends weak equivalences to isomorphisms. Now we have to show that γ_C has the universal property of the localization. Let $F : C \to D$ be a functor carrying weak equivalences to isomorphisms. We would like to find a functor $F' : \mathrm{Ho}(C) \to D$ with the property that $F'\gamma_C = F$. Since γ_C is the identity on objects, this equality defines F' on the objects of $\mathrm{Ho}(C)$. By proposition (3.29) again, defining F' on the image of morphisms of C by γ_C completely determines it (F' must send the inverse of the images by γ_C of weak equivalences, to the inverse of their images by F'). Thus F', if it exists, is uniquely determined.

To ensure that F' is well-defined, we give an explicit expression: by the proof of proposition (3.29), a map $\theta \in \mathrm{Ho}(C)(X, Y)$ admits the representation $\gamma_C(p_Y)\gamma_C(\iota_{QY})^{-1}\gamma_C(f)\gamma_C(\iota_{QX})\gamma_C(p_X)^{-1}$ for some map $f : RQX \to RQY$ we defined up to homotopy. The condition $F'\gamma_C = F$ and functoriality of F' forces us to set:

$$F'(\theta) = F(p_Y)F(\iota_{QY})^{-1}F(f)F(\iota_{QX})F(p_X)^{-1}$$

By the claim below, F' is well defined : the map f was defined up to homotopy (actual homotopy and not only with the generated equivalence relation) and the images by F of left-homotopic maps are equal. Also F sends weak equivalences to isomorphisms by hypothesis.

Moreover, if we had $\theta = \gamma_C(g)$ for some $g \in C(X, Y)$, notice that

$$\begin{aligned}\gamma_C(p_Y)\gamma_C(\iota_{QY})^{-1}\gamma_C(\bar{\tilde{g}})\gamma_C(\iota_{QX})\gamma_C(p_X)^{-1} &= \gamma_C(p_Y)\gamma_C(\iota_{QY})^{-1}\gamma_C(\iota_{QY})\gamma_C(\tilde{g})\gamma_C(p_X)^{-1} \\ &= \gamma_C(g)\gamma_C(p_X)\gamma_C(p_X)^{-1} = \gamma_C(g)\end{aligned}$$

hence $f = \bar{\tilde{g}}$ satisfies the required conditions, so

$$\begin{aligned}F'(\gamma_C(g)) &= F(p_y)F(\iota_{QY})^{-1}F(\bar{\tilde{g}})F(\iota_{QX})F(p_X)^{-1} \\ &= F(p_Y)F(\iota_{QY})^{-1}F(\iota_{QY})F(\tilde{g})F(p_X)^{-1} \\ &= F(g)F(p_X)F(p_X)^{-1} \\ &= F(g)\end{aligned}$$

Hence $F'\gamma_C = F$.

It only remains to check functoriality of F'. For the identities: let $\mathrm{id}_X^{\mathrm{Ho}(C)} \in \mathrm{Ho}(C)(X, X)$, we have $\gamma_C(\mathrm{id}_X^C) = \mathrm{id}_X^{\mathrm{Ho}(C)}$ so $F'(\mathrm{id}_X^{\mathrm{Ho}(C)}) = F'(\gamma_C(\mathrm{id}_X^C)) = F(\mathrm{id}_X^C) = \mathrm{id}_{F(X)}^D = \mathrm{id}_{F'(\gamma_C(X))}^D = \mathrm{id}_{F'X}^D$. For composition, let $\theta \in \mathrm{Ho}(C)(X, Y)$ and $\theta' \in \mathrm{Ho}(C)(Y, Z)$. Choose representatives f and f' as in the proof of proposition (3.29). Then $f' \circ f$ satisfies the same condition for $\theta' \circ \theta$: indeed,

$$\begin{aligned}[f' \circ f] = [f'][f] &= \gamma_C(\iota_{QZ})\gamma_C(p_Z)^{-1}\theta'\gamma_C(p_Y)\gamma_C(\iota_{QY})^{-1}\gamma_C(\iota_{QY})\gamma_C(p_Y)^{-1}\theta\gamma_C(p_X)\gamma_C(\iota_{QX})^{-1} \\ &= \gamma_C(\iota_{QZ})\gamma_C(p_Z)^{-1} \circ (\theta'\theta) \circ \gamma_C(p_X)\gamma_C(\iota_{QX})^{-1}\end{aligned}$$

We get:

$$\begin{aligned}F'(\theta')F'(\theta) &= F(p_Z)F(\iota_{QZ})^{-1}F(f')F(\iota_{QY})F(p_Y)^{-1}F(p_Y)F(\iota_{QY})^{-1}F(f)F(\iota_{QX})F(p_X)^{-1} \\ &= F(p_Z)F(\iota_{QZ})^{-1}F(f'f)F(\iota_{QX})F(p_X)^{-1} = F'(\theta'\theta)\end{aligned}$$

hence the functoriality.

Claim. The images by F of left homotopic maps are equal.

Proof of the claim. Indeed, let f and f' in $C(A, B)$ be left homotopic. Choose a homotopy H from f to f' through a cylinder object $A \amalg A \xrightarrow{\iota_0+\iota_1} A \wedge I \xrightarrow[\sim]{p} A$ for A. Then $p\iota_0 = \mathrm{id}_A^C = p\iota_1$, implying that $F(p)F(\iota_0) = \mathrm{id}_{F(A)}^D = F(p)F(\iota_1)$. But since p is a weak equivalence, $F(p)$ is an isomorphism: post-composing with its inverse in the previous equality, we get $F(\iota_0) = F(\iota_1)$. Then

$$F(f) = F(H\iota_0) = F(H)F(\iota_0) = F(H)F(\iota_1) = F(H\iota_1) = F(f')$$

as desired. □

Since definition (3.3) only depends on weak equivalences, this theorem tells us that the homotopy category, even when constructed via homotopy of maps, does not depend on the choice of fibrations and cofibrations. We will see in section (8) that different model structures may share the same class of weak equivalences. In this situation, their homotopy categories are the same.

4 A theorem about equivalence of homotopy categories

4.1 The problem

Now that we built homotopy categories, we would like to compare them for different model categories. Given a functor $F : C \to D$ of model categories, we would like to extend it into a functor $F' : \mathrm{Ho}(C) \to D$ or $F'' : \mathrm{Ho}(C) \to \mathrm{Ho}(D)$. In general there is no solution, since nothing guarantees that the functor F sends homotopic maps, even between cofibrant and fibrant objects, to homotopic maps, and even less to equal maps. An attempt at proving this would be by applying the functor F to a given homotopy H of maps in C. One of the problems that arise is that the image of a cylinder object for an object X in C is not necessarily a cylinder object for $F(X)$ in D: weak equivalences may not be preserved. However, as we will see in this section, we can solve this problem under reasonable additional assumptions, if we only ask for extension up to some natural transformation with a minimality/maximality property.

4.2 Derived functors

Before solving our problem, we need a rigorous definition of "best approximation up to a natural transformation". The answer is a construction called *derived functors*. Here is the definition:

Definition 4.1 (Left derived functor). Let C be a model category and $F : C \to D$ a functor (D needs not to be a model category). A *left derived functor* for F is a functor $LF : \mathrm{Ho}(C) \to D$ with the data of a natural transformation $\tau : LF \circ \gamma_C \to F$, which has the following universal property ("universal from the left"): for any other such pair of a functor G and natural transformation $\sigma : G \circ \gamma_C \to F$, there exists a unique natural transformation $\sigma' : G \to LF$ such that:

$$\begin{tikzcd} G \circ \gamma_C \arrow[r, "\sigma' \gamma_C"'] \arrow[rr, bend left, "\sigma"] & LF \circ \gamma_C \arrow[r, "\tau"'] & F \end{tikzcd}$$

($LF \circ \gamma_C$ is "closer" to F than $G \circ \gamma_C$).

Definition 4.2 (Right derived functor). Let C be a model category and $F : C \to D$ a functor (D needs not to be a model category). A *right derived functor* for F is a functor $RF : \mathrm{Ho}(C) \to D$ with the data of a natural transformation $\tau : F \to RF \circ \gamma_C$, which has the following universal property ("universal from the right"): for any other such pair of a functor G and natural transformation $\sigma : F \to G \circ \gamma_C$, there exists a unique natural transformation $\sigma' : RF \to G$ such that:

$$\begin{tikzcd} F \arrow[r, "\tau"'] \arrow[rr, bend left, "\sigma"] & RF \circ \gamma_C \arrow[r, "\sigma' \gamma_C"'] & G \circ \gamma_C \end{tikzcd}$$

(again $RF \circ \gamma_C$ is "closer" to F than $G \circ \gamma_C$).

Once more we have uniqueness up to natural isomorphism (as usual, suppose that LF, $L'F$ are two left-derived functors for F. By applying the universal property of the first to the second and vice versa, we get natural transformations σ' and σ'', whose compositions $\sigma'\sigma''$, respectively $\sigma''\sigma'$, make the diagram in the universal property of LF applied to itself, respectively $L'F$, commute, so by uniqueness these compositions are identities: LF and $L'F$ are naturally isomorphic).

This turns out to be the right notion to solve our first problem (extend the source to the homotopy category). To solve our second problem (extend both the source and the target to homotopy categories), we need another definition:

Definition 4.3 (Total left/right derived functor). Let C and D be model categories and consider a functor $F : C \to D$. A *total left* (respectively *right*) *derived functor* for F is a left (respectively right) derived functor $\mathbb{L}F : \mathrm{Ho}(C) \to \mathrm{Ho}(D)$, respectively $\mathbb{R}F : \mathrm{Ho}(C) \to \mathrm{Ho}(D)$, for the functor $\gamma_D \circ F$.

4.3 The theorem

Here is the advertised theorem, where an adjunction between model categories gives rises to an adjunction between their respective homotopy categories:

Theorem 4.4 (Adjunction between the homotopy categories). *Let C and D be model categories with an adjunction:*

$$\begin{tikzcd} F : C \arrow[r, shift left] \arrow[r, phantom, "\perp"] & D : G \arrow[l, shift left] \end{tikzcd}$$

If one of the following holds:

(i) the left adjoint F preserves cofibrations and acyclic cofibrations

(ii) the right adjoint G preserves fibrations and acyclic fibrations

(iii) F preserves cofibrations and G preserves fibrations

then the total left derived functor of F and the total right derived functor of G exist, and they form an adjoint pair:

$$\mathbb{L}F : \operatorname{Ho}(C) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \operatorname{Ho}(D) : \mathbb{R}G$$

In this situation, the adjunction $L \dashv R$ is called a Quillen adjunction. *Under the additional assumption that the natural bijections*

$$\alpha_{X,Y} : D(F(X), Y) \to C(X, G(Y)),\ f \mapsto f^\sharp$$

and

$$\beta_{X,Y} : C(X, G(Y)) \to D(F(X), Y),\ f \mapsto f^\flat$$

induced by the adjunction $F \dashv G$ both preserve weak equivalences for every cofibrant object X in C and fibrant object Y in D (equivalently, $\alpha_{X,Y}$ preserves and creates weak equivalences), the left and right total derived functors $\mathbb{L}F$ and $\mathbb{R}G$ form an equivalence of categories $\operatorname{Ho}(C) \simeq \operatorname{Ho}(D)$*. In this situation the Quillen adjunction $L \dashv R$ is also called a* Quillen equivalence.

Proof. We use the notation of the statement during the whole proof. The functors R and Q for the category D are denoted by R' and Q' to distinguish them from the functors R and Q for C.

Conditions (i), (ii) and (iii) are equivalent. We show $(iii) \implies (ii)$ and the other proofs are similar. We have to show that G preserves acyclic fibrations. Let $p : A \xrightarrow{\sim} B$ be an acyclic fibration in D. We want to show that $G(p)$ is an acyclic fibration. By proposition (2.9), it suffices to show that $G(p)$ has the RLP with respect to cofibrations. The first diagram below shows a lifting problem in D, and applying β we get the second diagram:

$$\begin{tikzcd} E \arrow[r, "k"] \arrow[d, hook, "\iota"'] & G(A) \arrow[d, "G(p)"] & & F(E) \arrow[r, "k^\flat"] \arrow[d, hook, "F(\iota)"'] & A \arrow[d, two heads, "p", "\wr"'] \\ E' \arrow[r, "h"'] & G(B) \arrow[r, phantom, "\overset{\beta}{\rightsquigarrow}"] & {} & F(E') \arrow[r, "h^\flat"'] \arrow[ur, dotted, "\exists \ell"] & B \end{tikzcd}$$

The lift ℓ exists because F preserves cofibrations by assumption. Commutativity of the square is preserved by naturality of β. Now we check that $\ell^\sharp : E' \to G(A)$ is a solution to our lifting problem. Indeed, α in a natural transformation between the functors $D(F(-), -)$ and $C(-, G(-))$, so naturality yields:

$$G(p)\ell^\sharp = (C(\mathrm{id}_{E'}^{\mathrm{op}}, G(p)) \circ \alpha_{E',A})(\ell) = (\alpha_{E',B} \circ D(F(\mathrm{id}_{E'})^{\mathrm{op}}, p))(\ell) = \alpha_{E',B}(p\ell) = \alpha_{E',B}(h^\flat) = h$$

$$\ell^\sharp \iota = (C(\iota^{\mathrm{op}}, G(\mathrm{id}_A)) \circ \alpha_{E',A})(\ell) = (\alpha_{E,A} \circ D(F(\iota)^{\mathrm{op}}, \mathrm{id}_A))(\ell) = \alpha_{E,A}(\ell F(\iota)) = \alpha_{E,A}(k^\flat) = k.$$

The derived functors exist. By definition, a total right derived functor $\mathbb{R}G$ for G is a right derived functor for $\gamma_C G$. Its source is $\operatorname{Ho}(D)$ and its target is $\operatorname{Ho}(C)$.

- We would like to define a candidate for $\mathbb{R}G$ by using the universal property of the localization, so we first need a functor $D \to \operatorname{Ho}(C)$ sending weak equivalences to isomorphisms. The composite $\gamma_C \circ G$ would be a great candidate, but a priori G does not preserves weak equivalences, so the composition $\gamma_C \circ G$ does not necessarily satisfy the condition we need. However, it turns out that G preserves weak equivalences between fibrant objects, as proved in lemma (4.5) below.

- Thus we make every object fibrant: we would like to precompose $\gamma_C \circ G$ by R', so we check that $\gamma_C G$ induces a functor $K : \pi D_f \to \operatorname{Ho}(C)$. Given fibrant objects A and B and $f, g : A \to B$ two maps in D with $[f]_\ell = [g]_\ell$, we have to show that $\gamma_C G(f) = \gamma_C G(g)$. If f is really left homotopic to g, we have $[Gf]_\ell = [Gg]_\ell$ by lemma (4.5) below. If f and g are related by a sequence $f \overset{\ell}{\sim} f_1 \overset{\ell}{\sim} \dots \overset{\ell}{\sim} g$ of homotopies, we apply the lemma to each homotopy and get the same result. The target GB of Gf and Gg is fibrant: G is a right adjoint so it preserves limits, in particular $G(*_D) = *_C$ so the image under G of the map $B \twoheadrightarrow *_D$ is the unique map $GB \to *_C$ and it is a fibration because G preserves fibrations. This implies by proposition (3.26) that $Q'Gf = Q'Gg$ (once more, if necessary, apply the lemma to a sequence of homotopies $Gf \overset{\ell}{\sim} g_1 \overset{\ell}{\sim} \dots \overset{\ell}{\sim} Gg$) so $R'Q'Gf = R'Q'Gg$ and $\gamma_C G(f) = \gamma_C G(g)$.

- Then, KR' sends weak equivalences to isomorphisms: by proposition (3.27), R' sends a weak equivalence to the left homotopy class of a weak equivalence between fibrant objects. The functor K sends this equivalence class to the image by $\gamma_C \circ G$ of any of its representatives. The functor G sends this particular representative to a weak equivalence by the lemma below, and γ_C sends it to an isomorphism by proposition (3.29) so in the end we get an isomorphism.

- By theorem (3.31), $\mathrm{Ho}(D)$ is the localization of D with respect to the class of weak equivalences. Applying the universal property of the localization, we obtain a functor $\mathbb{R}G : \mathrm{Ho}(D) \to \mathrm{Ho}(C)$ such that $\mathbb{R}G \circ \gamma_D = KR'$. There is a natural transformation $\tau : \gamma_C G \to \mathbb{R}G\gamma_D = KR'$ given by $\tau_A = \gamma_C(G(\iota_A)) \in \mathrm{Ho}(C)(GA, GR'A)$ for any object A of D. We check naturality: let $f \in D(A, B)$. The following diagram is commutative:

$$\begin{tikzcd} GA \arrow[r, "\gamma_C(G(\iota_A))"] \arrow[d, "\gamma_C(Gf)"'] & GR'A \arrow[d, "KR'f"] \\ GB \arrow[r, "\gamma_C(G(\iota_B))"'] & GR'B \end{tikzcd}$$

because $\bar{f}\iota_A = \iota_B f$ so

$$\gamma_C(G(\iota_B))\gamma_C(G(f)) = \gamma_C(G(\iota_B f)) = \gamma_C(G(\bar{f}\iota_A)) = \gamma_C(G(\bar{f}))\gamma_C(G(\iota_A)) = KR'f \circ \gamma_C(G(\iota_A))$$

by definition of K. Hence the naturality of τ.

- It only remains to show that the pair $(\mathbb{R}G, \tau)$ is universal from the right. Let (H, σ) be another pair with $H : \mathrm{Ho}(D) \to \mathrm{Ho}(C)$ and $\sigma : \gamma_C G \to H\gamma_D$. We have to show that there exists a unique natural transformation $\sigma' : \mathbb{R}G \to H$ such that $(\sigma'\gamma_D) \circ \tau = \sigma$. Suppose first that σ' is such a natural transformation. Let A be an object of D and consider applying all these natural transformations to the map $A \underset{\iota_A}{\overset{\sim}{\hookrightarrow}} R'A$:

$$\begin{tikzcd} \gamma_C(G(A)) \arrow[r, "\tau_A"] \arrow[d, "\gamma_C(G(\iota_A))"'] \arrow[rr, bend left, "\sigma_A"] & \mathbb{R}G(\gamma_D(A)) \arrow[r, "\sigma'_{\gamma_D(A)}"] \arrow[d, "\mathbb{R}G(\gamma_D(\iota_A))"'] & H(\gamma_D(A)) \arrow[d, "H(\gamma_D(\iota_A))"'] \\ \gamma_C(G(R'A)) \arrow[r, "\tau_{R'A}"] \arrow[rr, bend right, "\sigma_{R'A}"'] & \mathbb{R}G(\gamma_D(R'A)) \arrow[r, "\sigma'_{\gamma_D(R'A)}"] & H(\gamma_D(R'A)) \end{tikzcd}$$

We have $\mathbb{R}G(\gamma_D(R'A)) = KR'R'A = KR'A = GR'A = \mathbb{R}G(\gamma_D(A))$ and $\mathbb{R}G(\gamma_D(\iota_A)) = KR'\iota_A$. But $\overline{\iota_A}$ can be chosen to be $\mathrm{id}_{R'A}$ (because $R'R'A = R'A$) so we get the identity $\mathrm{id}_{GR'A}$. And $\tau_{R'A} = \gamma_C(G(\iota_{R'A}))$ is the identity too (because $\iota_{R'A} = \mathrm{id}_{R'A}$, see the construction of R' in proposition (3.27)). So the diagram becomes:

$$\begin{tikzcd} GA \arrow[r, "\gamma_C(G(\iota_A))"] \arrow[d, "\gamma_C(G(\iota_A))"'] \arrow[rr, bend left, "\sigma_A"] & KR'A \arrow[r, "\sigma'_{\gamma_D(A)}"] \arrow[d, equal] & H(\gamma_D(A)) \arrow[d, "H(\gamma_D(\iota_A))"'] \\ GR'A \arrow[r, equal] \arrow[rr, bend right, "\sigma_{R'A}"'] & GR'A \arrow[r, "\sigma'_{\gamma_D(R'A)}"] & H(\gamma_D(R'A)) \end{tikzcd}$$

so $H(\gamma_D(\iota_A)) \circ \sigma'_{\gamma_D(A)} = \sigma'_{\gamma_D(R'A)} = \sigma_{R'A}$ and since ι_A is a weak equivalence and γ_D sends weak equivalences to isomorphisms (see proposition (3.29)), $H(\gamma_D(\iota_A))$ is an isomorphism. Finally $\sigma'_{\gamma_D(A)} = H(\gamma_D(\iota_A))^{-1} \circ \sigma_{R'A}$. Since γ_D is the identity on objects, this suffices to determine entirely σ': hence the uniqueness. It remains to show that σ' defined in this way is indeed a natural transformation. Let $\gamma_D(A), \gamma_D(B)$ be objects in $\mathrm{Ho}(D)$ and consider a morphism

$\theta \in \mathrm{Ho}(D)(A, B)$. Then we want to show that:

$$\begin{tikzcd} \mathbb{R}G(\gamma_D(A)) \arrow[r, dashed, "\sigma'_{\gamma_D(A)}"] \arrow[d, "\mathbb{R}G(\theta)"] & H(\gamma_D(A)) \arrow[d, "H(\theta)"] \\ \mathbb{R}G(\gamma_D(B)) \arrow[r, dashed, "\sigma'_{\gamma_D(B)}"] & H(\gamma_D(B)) \end{tikzcd}$$

is commutative. For this, we apply proposition (3.29): the morphisms of $\mathrm{Ho}(D)$ are generated under composition by images by γ_D of morphisms in D and inverses of images by γ_D of weak equivalences. Thus, if we show that the diagram above is commutative when θ belongs to one of these types of maps, we are done (then, we compose commutative squares to show that the diagram we are interested in is commutative). If $\theta = \gamma_D(f)$ for some $f \in D(A, B)$, then by naturality of σ applied to the map $\overline{f}$:

$$\begin{aligned} H(\gamma_D(f)) \circ \sigma'_{\gamma_D(A)} &= H(\gamma_D(f)) \circ H(\gamma_D(\iota_A))^{-1} \circ \sigma_{R'A} \\ &= H(\gamma_D(f) \circ \gamma_D(\iota_A)^{-1}) \circ \sigma_{R'A} \\ &\overset{(\star)}{=} H(\gamma_D(\iota_B)^{-1} \circ \gamma_D(\overline{f})) \circ \sigma_{R'A} \\ &= H(\gamma_D(\iota_B)^{-1}) \circ H(\gamma_D(\overline{f})) \circ \sigma_{R'A} \\ &= H(\gamma_D(\iota_B))^{-1} \circ \sigma_{R'B} \circ \gamma_C(G(\overline{f})) \\ &= \sigma'_{\gamma_D(B)} \circ \mathbb{R}G(\gamma_D(f)) \end{aligned}$$

We justify $(\star)$: by definition of $\overline{f}$, $\overline{f}\iota_A = \iota_B f$ so $\gamma_D(\overline{f})\gamma_D(\iota_A) = \gamma_D(\iota_B)\gamma_D(f)$ implying that $\gamma_D(\iota_B)^{-1}\gamma_D(\overline{f}) = \gamma_D(f)\gamma_D(\iota_A)^{-1}$. The proof for the inverses of images of weak equivalences is the same (indeed $H(\gamma_D(w)^{-1}) = H(\gamma_D(w))^{-1}$ for any weak equivalence w in D).

- Finally, $\mathbb{R}G$ as defined above is a total right derived functor for G.

The proof for the existence of $\mathbb{L}F$ is dual to the proof we just did, since the concepts of right and left derived functors are dual and also the properties of F are dual to the properties of G we used (G preserves acyclic fibrations, fibrations and limits whereas F preserves acyclic cofibrations, cofibrations and colimits).

They are adjoint functors. To show that the functors $\mathbb{L}F$ and $\mathbb{R}G$ obtained in the last paragraph form an adjoint pair, we have to establish a natural bijection between $\mathrm{Ho}(D)(\mathbb{L}F(X), Y)$ and $\mathrm{Ho}(C)(X, \mathbb{R}G(Y))$ for any objects X in $\mathrm{Ho}(C)$ and Y in $\mathrm{Ho}(D)$. While proving the existence of $\mathbb{R}G$ in the last paragraph, we saw that $\mathbb{R}G(Y) = GR'Y$. Dually $\mathbb{L}F(X) = FQX$. Hence we can construct a bijection $\mathbb{A}_{X,Y}$:

$$\begin{tikzcd} \mathrm{Ho}(D)(\mathbb{L}F(X), Y) = \mathrm{Ho}(D)(FQX, Y) \arrow[d, "\text{post-composition by } \gamma_D(\iota_Y)"] \\ \mathrm{Ho}(D)(FQX, R'Y) \arrow[d, "\tilde{\alpha}_{QX,R'Y}"] \\ \mathrm{Ho}(C)(QX, GR'Y) \arrow[d, "\text{pre-composition by } \gamma_C(p_X)^{-1}"] \\ \mathrm{Ho}(C)(X, GR'Y) = \mathrm{Ho}(C)(X, \mathbb{R}G(Y)) \end{tikzcd}$$

The first and last map are bijections because ι_Y and p_X are weak equivalences, so their image by γ_D is an isomorphism, and pre/post-composing by their respective inverses gives inverse maps. For the second map, we want to show that $\alpha_{QX,R'Y} : D(FQX, R'Y) \to C(QX, GR'Y)$ induces a bijection $\tilde{\alpha}_{QX,R'Y} : \mathrm{Ho}(D)(FQX, R'Y) \to \mathrm{Ho}(C)(QX, GR'Y)$. Because FQX and QX are cofibrant (in a similar way that G preserves fibrant objects, F preserves cofibrant ones), and

$R'Y$, $GR'Y$ are fibrant, we know by lemma (3.30) that $\mathrm{Ho}(D)(FQX, R'Y) \cong \pi(FQX, R'Y)$ and $\mathrm{Ho}(C)(QX, GR'Y) \cong \pi(QX, GR'Y)$. So it suffices to show that if $f, g \in D(FQX, R'Y)$ are homotopic, so are $\alpha_{QX,R'Y}(f) = f^\sharp$ and $\alpha_{QX,R'Y}(g) = g^\sharp$. Let $R'Y \xrightarrow[\sim]{\iota} (R'Y)^I \overset{p}{\twoheadrightarrow} R'Y \times R'Y$ be a good path object for $R'Y$ in D, and $H : FQX \to (R'Y)^I$ be a good homotopy between f and g. The object $(R'Y)^I$ is fibrant: the unique map $(R'Y)^I \to *_D$ can be obtained as the composition of a projection $(R'Y)^I \overset{p}{\twoheadrightarrow} R'Y \times R'Y \to R'Y$, which is a fibration by the dual of remark (3.8), and the unique map $R'Y \twoheadrightarrow *_D$. Since G preserves weak equivalences between fibrant objects by lemma (4.5), fibrations by hypothesis, and limits as a right adjoint,

$$GR'Y \xrightarrow[\sim]{G\iota} G((R'Y)^I) \overset{Gp}{\twoheadrightarrow} G(R'Y \times R'Y) = GR'Y \times GR'Y$$

is a good path object for $GR'Y$ in C, and $H^\sharp = QX \to G((R'Y)^I)$ is a homotopy between $f^\sharp$ and $g^\sharp$:

$$\begin{aligned} G(p)H^\sharp &= C(\mathrm{id}_{QX}^{\mathrm{op}}, G(p))(\alpha_{QX,(R'Y)^I}(H)) = \alpha_{QX,R'Y\times R'Y}(D(F(\mathrm{id}_{QX})^{\mathrm{op}}, p)(H)) \quad \text{by naturality of } \alpha \\ &= (pH)^\sharp = (f,g)^\sharp \quad \text{by definition of } H \\ &= (f^\sharp, g^\sharp) \end{aligned}$$

by uniqueness in the universal property of the product: indeed, if $R'Y \xleftarrow{\pi_0} R'Y \times R'Y \xrightarrow{\pi_1} R'Y$ is the product, we know that $GR'Y \xleftarrow{G\pi_0} GR'Y \times GR'Y \xrightarrow{G\pi_1} GR'Y$ is also a product, so that $G(\pi_0) \circ (f,g)^\sharp = (\pi_0(f,g))^\sharp = f^\sharp$ and $G(\pi_1) \circ (f,g)^\sharp = (\pi_1(f,g))^\sharp = g^\sharp$. Hence $f^\sharp \overset{r}{\sim} g^\sharp$ as desired.

The map $\mathbb{A}_{X,Y}$ is a bijection because in a similar way β induces a map $\mathbb{B}_{X,Y}$ in the other direction, which is an inverse for $\mathbb{A}_{X,Y}$: the latter was induced by α.

Naturality comes from the fact that, as pre- and post- composition by maps, the first and third bijections defined above are natural in X and Y, and also the map induced by α is natural in X and Y because α is a natural transformation.

Finally, we obtain an adjunction $\mathbb{L}F \dashv \mathbb{R}G$.

Additional property. Under the additional assumption of the theorem, we show that the unit and the co-unit associated to the adjunction found in the previous paragraph are in fact natural isomorphisms. This will imply that $\mathbb{L}F \circ \mathbb{R}G \cong \mathrm{id}_{\mathrm{Ho}(D)}$ and $\mathbb{R}G \circ \mathbb{L}F \cong \mathrm{id}_{\mathrm{Ho}(C)}$ so $\mathbb{L}F$ and $\mathbb{R}G$ are inverse equivalences of categories between $\mathrm{Ho}(C)$ and $\mathrm{Ho}(D)$. The unit η associated to the adjunction is given by $\eta_X = \mathbb{A}_{X,\mathbb{L}FX}(\mathrm{id}_{\mathbb{L}FX})$ for any object X in $\mathrm{Ho}(C)$ and the co-unit ε is given by $\varepsilon_Y = \mathbb{B}_{\mathbb{R}GY,Y}(\mathrm{id}_{\mathbb{R}GY})$ for any object Y in $\mathrm{Ho}(D)$. Let X in $\mathrm{Ho}(C)$, we have to show that η_X is an isomorphism. We compute: $\eta_X = \mathbb{A}_{X,\mathbb{L}FX}(\mathrm{id}_{\mathbb{L}FX}) = \tilde{\alpha}_{QX,R'\mathbb{L}FX}(\gamma_D(\iota_{\mathbb{L}FX}))\gamma_C(p_X)^{-1}$.

By definition $\iota_{\mathbb{L}FX}$ is a weak equivalence, so under α it is sent to a weak equivalence by hypothesis, and when passing to the homotopy class by γ_C, we get an isomorphism, so in the end η_X is an isomorphism and η is a natural isomorphism. The argument for ε is the same, and this finishes the proof. ∎

Lemma 4.5. *In the situation of theorem (4.4), G preserves weak equivalences between fibrant objects. Moreover, if f and g are left homotopic maps between fibrant objects, $Gf = Gg$.*

Proof. Let A, B in D be fibrant objects and $f : A \xrightarrow{\sim} B$. A clever trick suggested in the article by Dwyer and Spaliński (1995) (cf Lemma 9.9 p44) is to factor the map $(\mathrm{id}_A, f) : A \longrightarrow A \times B$ using axiom MC5. We have the following situation:

$$\begin{tikzcd} & & A \\ A \arrow[r, "{(\mathrm{id}_A, f)}"] \arrow[d, hook, "q", "\wr"'] & A \times B \arrow[ur, "\pi_A"] \arrow[dr, two heads, "\pi_B"] & \\ C \arrow[ur, two heads, "p"'] & & B \end{tikzcd}$$

Since A and B are fibrant, the compositions $\pi_A p$ and $\pi_B p$ are fibrations (with the same proof as the dual of remark (3.8)). Because $(\pi_A p) \circ q = \mathrm{id}_A$ is a weak equivalence and q too, the 2 of 3 rule implies that $\pi_A p$ is an acyclic fibration. Similarly, $(\pi_B p) \circ q = f$ and q are weak equivalences so $\pi_B p$ is an acylic fibration too. By hypothesis G preserves acyclic fibrations, hence $G(\pi_A p)$ and $G(\pi_B p)$

are acyclic fibrations, in particular they are weak equivalences. We apply the 2 of 3 rule once more to $G(\pi_A p)$, $G(q)$ and $G(\pi_A p) \circ G(q) = G(\mathrm{id}_A) = \mathrm{id}_{G(A)}$ to obtain that $G(q)$ is a weak equivalence, and by composition of weak equivalences $G(f) = G(\pi_B p) \circ G(q)$ is a weak equivalence, as desired.

For the second part of the statement, consider $f, g : X \to Y$ maps in D with X and Y fibrant, and $f \overset{\ell}{\sim} g$. There exists a good cylinder object $X \amalg X \xhookrightarrow{\iota} X \wedge I \xrightarrow[\sim]{p} X$, for X in D and a good homotopy $H : X \wedge I \to Y$ between f and g. We upgrade this homotopy to a very good homotopy: by MC5 applied to p there is a factorization:

$$\begin{tikzcd} X \amalg X \arrow[r, hook, "\iota"] \arrow[rrr, bend right, "\mathrm{id}_X + \mathrm{id}_X"'] & X \wedge I \arrow[r, hook, "\iota'"] \arrow[rr, bend left, "p", "\sim"'] & (X \wedge I)' \arrow[r, two heads, "p'", "\sim"'] & X \end{tikzcd}$$

and by the 2 of 3 rule, ι' is a weak equivalence. So $(X \wedge I)'$ is a very good cylinder object for X, and by finding a lift H' in:

$$\begin{tikzcd} X \wedge I \arrow[r, "H"] \arrow[d, hook, "\iota'"', "\wr"] & Y \arrow[d, two heads] \\ (X \wedge I)' \arrow[r] \arrow[ur, dashed, "H'"] & * \end{tikzcd}$$

(Y is fibrant by hypothesis), we get a very good homotopy between f and g through $(X \wedge I)'$: indeed $H' \circ \iota' \iota = H\iota = f + g$.

So without loss of generality we may assume that p was a fibration at the beginning. The unique map $X \wedge I \to *_D$ is obtained as the composition of p and the unique map $X \twoheadrightarrow *_D$, hence it is a fibration. Because G preserves acyclic fibrations, $G(p)$ is an acyclic fibration. Hence if we write $\iota = \iota_0 + \iota_1$, the diagram:

$$GX \amalg GX \xrightarrow{G(\iota_0)+G(\iota_1)} G(X \wedge I) \overset{G(p)}{\underset{\sim}{\twoheadrightarrow}} GX,$$

is a cylinder object for GX in C, and $G(H) : G(X \wedge I) \to GY$ is a (left) homotopy between $G(f)$ and $G(g)$: indeed,

$$G(H) \circ (G(\iota_0) + G(\iota_1)) = G(H)G(\iota_0) + G(H)G(\iota_1) = G(H\iota_0) + G(H\iota_1) = G(f) + G(g).$$

This concludes the proof. ☐

5 A model category structure for topological spaces

As discussed in example (2.5), we might put a model structure on **Top** by localizing it with respect to the class of homotopy equivalences. This is called the *Hurewicz* or *Strøm model structure*. We now build our first example of a model category by choosing instead *weak* homotopy equivalences as weak equivalences. In this section, "homotopy" without further precision is taken in the usual sense of homotopy of topological maps and topological spaces. This model structure was defined by Quillen (1967).

5.1 Definition of the Quillen model structure and first properties

Theorem 5.1 (The Quillen model structure on **Top**). *Consider the category* **Top** *with objects the topological spaces and arrows the continuous maps of topological spaces. Define:*

- **We(Top)** *as the class of weak homotopy equivalences between two spaces, i.e. continuous maps inducing isomorphisms on the n-th homotopy groups for $n \geq 1$, and a bijection on the sets of path-connected components.*
- **Fib(Top)** *as the class of Serre fibrations, i.e. continuous maps $p : X \to Y$ having the RLP with respect to all inclusions $A \times \{0\} \to A \times I$ where A is a CW-complex.*
- **Cofib(Top)** *as the class of continuous maps having the LLP with respect to all maps that are both Serre fibrations and weak homotopy equivalences.*

This choice endows **Top** *with a model category structure, called the* Quillen model structure.

We will spend the rest of this section proving the theorem. As a warm-up, we begin with axioms MC0 and MC2:

- π_n is a functor from the category of pointed topological spaces to the category of groups for any $n \geq 1$ and π_0 is a functor from the category of topological spaces to the category of sets. Therefore, for any topological space X, the identity id_X induces the identity on all homotopy groups and on the set of path components, so id_X is a weak homotopy equivalence. Moreover, the 2 of 3 rule MC2 holds, in particular, the class of weak homotopy equivalences is stable under composition. Indeed, let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. They induce homomorphisms f_n and g_n on the π_n-groups for all $n \geq 1$ and maps f_0 and g_0 on the sets of path components. By functoriality, $g \circ f$ induces the maps $\{g_n \circ f_n\}_{n \in \mathbb{N}}$. The 2 of 3 rule holds for bijections of sets and group isomorphisms (if two maps among f_0, g_0, and $g_0 \circ f_0$ are bijections, so is the third: if f_0 and g_0 are bijections, then $g_0 \circ f_0$ is a bijection too by composition, if f_0 and $g_0 \circ f_0$ are bijections, then $g_0 = (g_0 \circ f_0) \circ f_0^{-1}$ is a bijection too, and the third case is symmetric). By applying it for each $n \in \mathbb{N}$ to the maps f_n, g_n and $g_n \circ f_n$, we get that if two maps among f, g, and $g \circ f$ are weak homotopy equivalences, then so is the third.
- The stability under composition of the class of Serre fibrations and the fact that it contains the identity follows directly from the dual of proposition (2.10) (with S the class of inclusions of CW-complexes into their cylinder).
- Cofibrations are also defined by a lifting property with respect to a fixed class of maps, so the same proposition implies stability under composition and that the identity is a cofibration.

To prove that the other axioms hold, we need a small lemma about lifting properties in **Top**, and to know a bit more about Serre fibrations:

Lemma 5.2 (Lifts for disjoint unions)**.** *Suppose that $\{f_i : X_i \to Y_i\}_{i \in I'}$ is a family of continuous maps (indexed by some set I'), all having the LLP with respect to a fixed class of continuous maps L. Then the map $\coprod_{i \in I'} f_i : \coprod_{i \in I'} X_i \to \coprod_{i \in I'} Y_i$ has the LLP with respect to all maps of L.*

Proof. The (continuous) map $\coprod_{i \in I'} f_i$ sends (i, x) (with $i \in I'$ and $x \in X_i$) to $(i, f_i(x))$. For all $i \in I'$, let $\iota_i : X_i \to \coprod_{n \in I'} X_n$ and $j_i : Y_i \to \coprod_{n \in I'} Y_n$ be the natural inclusions. Consider a map ℓ in L and a lifting problem (first diagram):

$$\begin{tikzcd} \coprod_{i \in I'} X_i \arrow[r, "h=\sum_{i \in I'} h\iota_i"] \arrow[d, "\coprod_{i \in I'} f_i"'] & A \arrow[d, "\ell"] \\ \coprod_{i \in I'} Y_i \arrow[r, "k=\sum_{i \in I'} kj_i"'] & B \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[r, "h\iota_i"] \arrow[d, "f_i"'] & A \arrow[d, "\ell"] \\ Y_i \arrow[r, "kj_i"'] & B \end{tikzcd}$$

By hypothesis, a lift $g_i : Y_i \to A$ exists in the second diagram for each $i \in I'$. The map $g = \sum_{i \in I'} g_i : \coprod_{i \in I'} Y_i \to A$ solves our first lifting problem: $g\left(\coprod_{i \in I'} f_i\right) = \sum_{i \in I'} g_i f_i = \sum_{i \in I'} h\iota_i = h$ and $\ell g = \sum_{i \in I'} \ell g_i = \sum_{i \in I'} kj_i = k$. $\square$

Lemma 5.3 (Serre fibration criterion)**.** *Let $p : X \to Y$ be a continuous map. If p has the RLP with respect to the inclusions $\{m_n : \mathbb{D}^n \times \{0\} \to \mathbb{D}^n \times I\}_{n \in \mathbb{N}}$, then p is a Serre fibration.*

Proof. This proof is inspired by lectures notes from Mitchell (2001) (see the bibliography (9) for more details, this is Theorem 2.3). We want to show that p has the RLP with respect to any inclusion $A \times \{0\} \to A \times I$ for A a CW-complex.

Firstly, p has the RLP with respect to the inclusions $\{m''_n : (\mathbb{D}^n \times \{0\}) \cup (\mathbb{S}^{n-1} \times I) \to \mathbb{D}^n \times I\}_{n \in \mathbb{N}}$. Let $n \in \mathbb{N}$. Note that $(\mathbb{D}^n \times \{0\}) \cup (\mathbb{S}^{n-1} \times I)$ is homeomorphic to $\mathbb{D}^n \times \{0\}$: a homeomorphism $\phi : (\mathbb{D}^n \times \{0\}) \cup (\mathbb{S}^{n-1} \times I) \to \mathbb{D}^n \times \{0\}$ shrinks the standard disk into the disk of radius 1/2 centered at 0 and sends $(x, t) \in \mathbb{S}^{n-1} \times I$ to $((1+t)/2)x$ ("flattens" $\mathbb{S}^{n-1} \times I$ onto the "shell" of interior radius 1/2 and exterior radius 1 in $\mathbb{D}^n$. In the case $n = 2$, it corresponds to an annulus of inner radius 1/2 and outer radius 1 in the plane). To turn a lift with respect to m_n into a lift with respect to m''_n, we choose a homeomorphism $\psi : \mathbb{D}^n \times I \to \mathbb{D}^n \times I$ such that $\psi \circ m''_n = m_n \circ \phi$. Given a lifting problem as in the first diagram below, we find a lift ℓ in the second diagram, and then $\ell \circ \psi$ is a lift in the original diagram since $p\ell\psi = k'\psi^{-1}\psi = k'$ and $\ell\psi m''_n = \ell m_n \phi = h'\phi^{-1}\phi = h'$.

$$\begin{tikzcd}
(\mathbb{D}^n \times \{0\}) \cup (\mathbb{S}^{n-1} \times I) \arrow[r, "h'"] \arrow[d, "m''_n"'] & X \arrow[d, "p"] \\
\mathbb{D}^n \times I \arrow[r, "k'"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}^n \times \{0\} \arrow[r, "h'\circ\phi^{-1}"] \arrow[d, "m_n"'] & X \arrow[d, "p"] \\
\mathbb{D}^n \times I \arrow[r, "k'\circ\psi^{-1}"'] \arrow[ur, dashed, "\exists \ell"] & Y
\end{tikzcd}$$

Let A be a CW complex and consider a lifting problem:

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "h"] \arrow[d, "\iota"'] & X \arrow[d, "p"] \\
A \times I \arrow[r, "k"'] & Y
\end{tikzcd}$$

A is obtained by successively attaching cells of increasing dimension: A can be written as an infinite union $\bigcup_{n\in\mathbb{N}} A_n$ (a sequential colimit with maps $a_n : A_n \to A_{n+1}$ and $\iota_n : A_n \to A$) where A_n is obtained from A_{n-1} by attaching n-cells (and A_0 is a set of points, namely 0-cells): for all $n \in \mathbb{N}$, there exists a set C_n containing maps $\mathbb{S}^{n-1} \to A_{n-1}$, such that:

$$\begin{tikzcd}
\coprod_{f\in C_n} \mathbb{S}^{n-1} \arrow[r, "\sum_{f\in C_n} f"] \arrow[d, "\coprod_{f\in C_n} m'_n"'] & A_{n-1} \arrow[d, "a_{n-1}"] \\
\coprod_{f\in C_n} \mathbb{D}^n \arrow[r, "b_{n-1}"'] \arrow[ur, phantom, "\ulcorner", very near end] & A_n
\end{tikzcd} \tag{1}$$

with $m'_n : \mathbb{S}^{n-1} \to \mathbb{D}^n$ the inclusion. We build a lift by induction on the dimension of the cells. The inclusion $A_0 \times \{0\} \to A_0 \times I$ is just a disjoint union of inclusions of a point into an interval (copies of the map m_0). By lemma (5.2), there exists a lift ℓ_0 in the diagram:

$$\begin{tikzcd}
A_0 \times \{0\} \arrow[r, "\iota_0"] \arrow[d, "\coprod_{x\in A_0} m_0"'] & A \times \{0\} \arrow[r, "h"] & X \arrow[d, "p"] \\
A_0 \times I \arrow[r, "\iota_0 \times \mathrm{id}_I"'] \arrow[urr, dashed, "\exists \ell_0"] & A \times I \arrow[r, "k"'] & Y
\end{tikzcd}$$

Fix $n \in \mathbb{N}$ and assume that we have already defined a lift ℓ_i in the diagram above with A_0 and ι_0 replaced by A_i and ι_i respectively for all $0 \le i \le n$, and the inclusion on the left-hand side, such that $\ell_j \circ (a_{j-1} \times \mathrm{id}_I) = \ell_{j-1}$ for all $1 \le j \le n$. There exists a lift ℓ_{n+1} in the diagram:

$$\begin{tikzcd}
(A_{n+1} \times \{0\}) \cup (A_n \times I) \arrow[r, "(h\circ\iota_{n+1})\cup\ell_n"] \arrow[d] & X \arrow[d, "p"] \\
A_{n+1} \times I \arrow[r, "k\circ\iota_{n+1}"'] \arrow[ur, dashed, "\exists \ell_{n+1}"] & Y
\end{tikzcd}$$

(by hypothesis ℓ_n and $h \circ \iota_{n+1}$ coincide on $a_n(A_n) \times \{0\}$).

Such a lift exists because the map on the left-hand side of the diagram is obtained as the pushout:

$$\begin{tikzcd}
\coprod_{f\in C_{n+1}} \big((\mathbb{D}^{n+1} \times \{0\}) \cup (\mathbb{S}^n \times I)\big) \arrow[r, "u := b_n \cup \left(\left(\coprod_{f\in C_{n+1}} f\right)\times \mathrm{id}_I\right)"] \arrow[d, "\coprod_{f\in C_{n+1}} m''_{n+1}"'] & (A_{n+1} \times \{0\}) \cup (A_n \times I) \arrow[d] \\
\coprod_{f\in C_{n+1}} (\mathbb{D}^{n+1} \times I) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A_{n+1} \times I
\end{tikzcd} \tag{2}$$

and by proposition (2.10), having the LLP with respect to p is a property that is stable by cobase change (and disjoint union by lemma (5.2)), and each m''_{n+1} has the LLP with respect to p as we saw above. Let us finish the proof before checking that (2) is a pushout. By construction we have $\ell_{n+1} \circ (a_n \times \mathrm{id}_I) = \ell_n$. Taking the product with I defines a functor from **Top** to itself, which is a left adjoint and preserves colimits. Then, by the universal property of the colimit, the maps $\{\ell_n\}_{n\in\mathbb{N}}$ induce a map $\bigcup_{n\in\mathbb{N}}(A_n \times I) \cong A \times I \to X$ which is the solution to our original lifting problem.

It only remains to show that the diagram (2) is a pushout. We have the following situation:

$$
\begin{tikzcd}
\coprod_{f\in C_{n+1}}(\mathbb{S}^n\times\{0\}) \arrow[r] \arrow[d] & \coprod_{f\in C_{n+1}}(\mathbb{S}^n\times I) \arrow[r, "\left(\sum_{f\in C_{n+1}} f\right)\times I"] \arrow[d] & A_n\times I \arrow[d] \\
\coprod_{f\in C_{n+1}}(\mathbb{D}^{n+1}\times\{0\}) \arrow[r] & \coprod_{f\in C_{n+1}}\left((\mathbb{D}^{n+1}\times\{0\})\cup(\mathbb{S}^n\times I)\right) \arrow[r, "u"] \arrow[d, "\coprod_{f\in C_{n+1}} m''_{n+1}"'] & (A_{n+1}\times\{0\})\cup(A_n\times I) \arrow[d] \\
& \coprod_{f\in C_{n+1}}(\mathbb{D}^{n+1}\times I) \arrow[r] & A_{n+1}\times I
\end{tikzcd}
$$

The square we are interested is the lower right one. It is a pushout because the blue rectangle (by applying the functor $-\times I$ to (1)) and the upper right square are pushouts. For the upper right square, it comes from the fact that both the upper left square (simply as "union" of sets) and the green rectangle are pushouts. For the green rectangle, it comes from the fact that, in the diagram below, the two triangles are commutative and the two squares in the middle are pushouts (the first one is (1) and the second one is simply a 'union" of sets), so the big rectangle is a pushout too:

$$
\begin{tikzcd}
& \coprod_{f\in C_{n+1}}(\mathbb{S}^n\times I) \arrow[dr, "\left(\sum_{f\in C_{n+1}} f\right)\times I"] & \\
\coprod_{f\in C_{n+1}}(\mathbb{S}^n\times\{0\}) \arrow[ur] \arrow[r, "\sum_{f\in C_{n+1}} f"'] \arrow[d] & A_n\times\{0\} \arrow[r] \arrow[d] & A_n\times I \arrow[d] \\
\coprod_{f\in C_{n+1}}(\mathbb{D}^{n+1}\times\{0\}) \arrow[r, "b_n"] \arrow[dr] & A_{n+1}\times\{0\} \arrow[r] \arrow[ul, phantom, "\ulcorner"] & (A_{n+1}\times\{0\})\cup(A_n\times I) \arrow[ul, phantom, "\ulcorner"] \\
& \coprod_{f\in C_{n+1}}\left((\mathbb{D}^{n+1}\times\{0\})\cup(\mathbb{S}^n\times I)\right) \arrow[ur, "u"'] &
\end{tikzcd}
$$

$\square$

Lemma 5.4 (Acyclic Serre fibration criterion)**.** *Let $p : X \to Y$ be a map. If p has the RLP with respect to the inclusions $\{m'_n : \mathbb{S}^{n-1} \to \mathbb{D}^n\}_{n\in\mathbb{N}}$, then p is a Serre fibration and a weak equivalence.*

Proof. To show that p is a Serre fibration we check that the hypotheses of lemma (5.3) hold. Let $n \in \mathbb{N}$. The inclusion $m_n : \mathbb{D}^n \times \{0\} \to \mathbb{D}^n \times I$ is obtained as:

$$
\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "m'_{n+1}"] \arrow[d, "m'_n"'] & \mathbb{D}^n\times\{0\} \arrow[d, "a"] \arrow[dd, bend left, "m_n"] \\
\mathbb{D}^n \arrow[r] & \partial(\mathbb{D}^n\times I) \arrow[d, "b"] \arrow[ul, phantom, "\ulcorner"] \\
\mathbb{S}^n \arrow[ur, "f"] \arrow[d, "m'_{n+1}"'] & \\
\mathbb{D}^{n+1} \arrow[r] & \mathbb{D}^n\times I
\end{tikzcd}
$$

where f is a homeomorphism, and the identification $\mathbb{D}^n \cup_{\mathbb{S}^{n-1}} (\mathbb{D}^n \times \{0\}) \approx \partial(\mathbb{D}^n \times I)$ is the identity on $\mathbb{D}^n \times \{0\}$.

Let S be the class of maps having the LLP with respect to p. By hypothesis, m'_n, and m'_{n+1} are in S. By proposition (2.10), a and b are also in S as cobase changes of m'_n and m'_{n+1} respectively, and $m_n = b \circ a$ too by composition. Hence m_n has the LLP with respect to p: equivalently p has the RLP with respect to m_n. By arbitrarity of n, p is a Serre fibration.

Now we check that p is a weak equivalence: by hypothesis, for any $y \in Y$ a lift exists in:

$$\begin{tikzcd}
\mathbb{S}^{-1} = \emptyset \arrow[r] \arrow[d, "m_0'"'] & X \arrow[d, "p"] \\
\mathbb{D}^0 = \{*\} \arrow[r, "c_y"'] \arrow[ur, dashed, "\exists \ell_y"] & Y
\end{tikzcd}$$

with c_y sending the point $*$ to y. In particular, $\ell_y(*)$ is a preimage of y by p. Since y is arbitrary in Y, p is surjective, so it is surjective also on path components. For injectivity on path components: suppose that there is a path $\gamma : \mathbb{D}^1 \to Y$ between $p(x)$ and $p(x')$ for some $x, x' \in X$. We need to show that x, x' are in the same path component of X. We have a diagram:

$$\begin{tikzcd}
\mathbb{S}^0 = \{-1, 1\} \arrow[r, "-1 \mapsto x, 1 \mapsto x'"] \arrow[d, "m_1'"'] & X \arrow[d, "p"] \\
\mathbb{D}^1 \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists \ell"] & Y
\end{tikzcd}$$

Then ℓ is a path between $\ell(-1) = \ell(m_1'(-1)) = x$ and $\ell(1) = \ell(m_1'(1)) = x'$ in X: x and x' belong to the same path component. So p induces a bijection between the respective sets of path components of X and Y.

Let $n \geq 1$ and $x_0 \in X$. Then $p_n^* : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is an isomorphism:

- It is injective: consider a map $\gamma : \mathbb{S}^n \to X$, and suppose that $p \circ \gamma$ is null-homotopic in Y. We have to show that γ was null-homotopic in the first place. There is a homotopy $H : \mathbb{S}^n \times I \to Y$ sending any point in $\mathbb{S}^n \times \{1\}$ to $p(x_0)$ and equal to $p \circ \gamma$ on $\mathbb{S}^n \times \{0\}$. Then H induces a map $\bar{H} : {(\mathbb{S}^n \times I)}\big/{(\mathbb{S}^n \times \{1\})} \approx \mathbb{D}^{n+1} \to Y$, where the homeomorphism ${(\mathbb{S}^n \times I)}\big/{(\mathbb{S}^n \times \{1\})} \approx \mathbb{D}^{n+1}$ sends $\mathbb{S}^n \times \{0\}$ to the boundary of the $(n+1)$-disk. By hypothesis, a lift exists in the following commutative diagram:

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, "\gamma"] \arrow[d, "m_{n+1}'"'] & X \arrow[d, "p"] \\
\mathbb{D}^{n+1} \arrow[r, "\bar{H}"'] \arrow[ur, dashed, "\exists \ell"] & Y
\end{tikzcd}$$

  Then ℓ precomposed by the quotient $\mathbb{S}^n \times I \to \mathbb{D}^{n+1}$ described above is a homotopy between γ and a constant map: on $\mathbb{S}^n \times \{0\}$ we get $\ell \circ m_{n+1}'$ which is exactly γ, and the image of any point in $\mathbb{S}^n \times \{1\}$ is $\ell(0)$ because $\mathbb{S}^n \times \{1\}$ is collapsed in one point in the quotient. Hence γ is null homotopic. Because p_n^* is a group homomorphism this suffices to prove injectivity.

- It is surjective: consider a pointed map $\Gamma : \mathbb{S}^n \to Y$. We would like to find $\gamma : \mathbb{S}^n \to X$ such that $p \circ \gamma$ is homotopic to Γ, so that $[\gamma] \in \pi_n(X, x_0)$ is a preimage for $[\Gamma] \in \pi_n(Y, p(x_0))$ by p_n^*. A lift exists in the square on the right-hand side:

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r] \arrow[d] & * \arrow[r, "c_{x_0}"] \arrow[d] & X \arrow[d, "p"] \\
\mathbb{D}^n \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathbb{S}^n \arrow[r, "\Gamma"'] \arrow[ur, dashed, "\exists \gamma"] & Y
\end{tikzcd}$$

  because the map $f : * \to \mathbb{S}^n$ is a cobase change of m_n' (as shown in the square on the left-hand side), so by proposition (2.10), p has the RLP with respect to f. In particular, $p \circ \gamma = \Gamma$.

- Therefore, p induces isomorphisms on all homotopy groups.

Finally, p is a weak equivalence. $\square$

Lemma 5.5 (Property of acyclic Serre fibrations)**.** *Let $p : X \to Y$ be a Serre fibration and a weak equivalence. Then p has the RLP with respect to all relative CW-inclusions.*

Proof. We will not prove this lemma here, for a proof see the article "The Quillen model category of topological spaces" by Hirschhorn (2019) (proposition 7.10 in the article implies our lemma. The proof uses homotopy fibers and long exact homotopy sequences). $\square$

5.2 Verification of the other axioms

5.2.1 Axiom MC1: limits and colimits

Small limits. Let D be a small category (representing the shape of the diagram we want to compute the limit of) and $F : D \to \mathbf{Top}$ a functor (a "realization" of the diagram in the category $\mathbf{Top}$). We show that the limit $\lim_D F$ exists in $\mathbf{Top}$ by direct construction. Since D is small, Ob(D) is a set. Consider $X = \prod_{d \in \mathrm{Ob}(D)} F(d)$ endowed with the product topology (a basis for this topology is given by the products of open subsets of $F(d)$ for all $d \in \mathrm{Ob}(D)$, with finitely many of them being different from the whole space). An element of X is of the form $(a_d)_{d \in \mathrm{Ob}(D)}$ with $a_d \in F(d)$ for all $d \in \mathrm{Ob}(D)$.

Consider the subset $Y = \{(a_d)_{d \in \mathrm{Ob}(D)} \in X \mid a_{d'} = F(f)(a_d)\ \forall f \in D(d,d')\}$, endowed with the subspace topology. For all $d \in \mathrm{Ob}(D)$, there is a map $\pi_d : Y \to F(d)$ sending $(a_{d'})_{d' \in \mathrm{Ob}(D)}$ to a_d. It is continuous: π_d is the canonical projection on the d-th coordinate. Moreover, for all $f \in D(d,d')$ we have:

$$\begin{tikzcd} Y \arrow[d, "\pi_d"'] \arrow[dr, "\pi_{d'}"] & \\ F(d) \arrow[r, "F(f)"'] & F(d') \end{tikzcd}$$

because for all $x = (a_{d'})_{d' \in \mathrm{Ob}(D)} \in Y$, we have $F(f)(\pi_d(x)) = F(f)(a_d) = a_{d'} = \pi_{d'}(x)$ by definition of Y. For the universal property, suppose that Z is a topological space with maps $q_d : Z \to F(d)$ and $F(f)q_d = q_{d'}$ for all objects d, d' in D and $f \in D(d,d')$. Then, there exists a map $q : Z \to Y$ given by $q(z) = (q_d(z))_{d \in \mathrm{Ob}(D)}$ for all $z \in Z$. By definition, $\pi_d \circ q = q_d$ for any $d \in \mathrm{Ob}(D)$. Hence the universal property is verified and $Y = \lim_D F$.

Small colimits. In the same setting, we show that the colimit $\mathrm{colim}_D F$ exists in $\mathbf{Top}$ by direct construction. Consider the set $X = \bigsqcup_{d \in \mathrm{Ob}(D)} F(d)$ endowed with the disjoint union topology ($A \subseteq X$ is open if and only if $A = \bigsqcup_{d \in \mathrm{Ob}(D)} A_d$ where $A_d \subseteq F(d)$ is open for all $d \in \mathrm{Ob}(D)$). Let $\sim$ be the equivalence relation on X generated by $(d,x) \sim (d', F(f)(x))$ for any $d, d' \in \mathrm{Ob(D)}$ and $f \in D(d,d')$. Consider $Y := X/_\sim$ with the quotient topology. We claim that Y is the desired colimit. Indeed, for any object d in D, there is a natural (continuous) map $i_d : F(d) \to Y$ given by the composition of the inclusion $F(d) \subseteq X$ and the quotient $X \to Y$. Moreover, for any map $f \in D(d,d')$, we have:

$$\begin{tikzcd} F(d) \arrow[r, "F(f)"] \arrow[d, "i_d"'] & F(d') \arrow[dl, "i_{d'}"] \\ Y = X/_\sim & \end{tikzcd}$$

since for all $x \in F(d)$, $i_{d'}(F(f)(x)) = (d', F(f)(x)) \sim (d,x) = i_d(x)$ by definition of $\sim$. For the universal property, consider any topological space Z with maps $\{j_d : F(d) \to Z\}_{d \in \mathrm{Ob}(D)}$ such that $j_{d'}(F(f)(x)) = j_d(x)\ \forall f \in D(d,d')$. Then, there is an induced map $\bar{j} : Y \to Z$ with $\bar{j} i_d = j_d$ for all $d \in \mathrm{Ob}(D)$: there is a map $j : X \to Z$ defined by j_d on each $F(d)$, and it passes to the quotient since $j(d,x) = j_d(x) = j_{d'}(F(f)(x)) = j(d', F(f)(x))$ for any $d, d' \in \mathrm{Ob}(D)$, $x \in F(d)$ and $f \in D(d,d')$. And for all $y \in F(d)$, $\bar{j} i_d(y) = j(d,y) = j_d(y)$. Hence the universal property is verified and $Y = \mathrm{colim}_D F$.

5.2.2 Axiom MC3: retracts

As for the verification of MC0, we use proposition (2.10): since Serre fibrations and cofibrations are defined using a lifting property with respect to a fixed class of maps (first we defined Serre fibrations, and then cofibrations using them), these classes of maps are stable by retracts.

Let $g : X \to Y$ be a weak homotopy equivalence, and suppose that $f : A \to B$ is a retract of g. We want to show that f is a weak equivalence. By definition there are continuous maps:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & X \arrow[r, "r_A"] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r, "\iota_B"'] & Y \arrow[r, "r_B"'] & B \end{tikzcd}$$

with $r_A\iota_A = \mathrm{id}_A$ and $r_B\iota_B = \mathrm{id}_B$. Let $n \geq 1$. For any basepoint $a_0 \in A$, by functoriality of π_n, this allows us to express $\pi_n(f) = f_n^*$ as a retract of $\pi_n(g) = g_n^*$:

$$\begin{tikzcd}
\pi_n(A,a_0) \arrow[r, "(\iota_A)_n^*"] \arrow[d, "f_n^*"'] & \pi_n(X,\iota_A(a_0)) \arrow[r, "(r_A)_n^*"] \arrow[d, "g_n^*"'] & \pi_n(A,a_0) \arrow[d, "f_n^*"'] \\
\pi_n(B,f(a_0)) \arrow[r, "(\iota_B)_n^*"'] & \pi_n(Y,g(\iota_A(a_0))) \arrow[r, "(r_B)_n^*"'] & \pi_n(B,f(a_0))
\end{tikzcd}$$

(indeed $(r_B)_n^* \circ (\iota_B)_n^* = (r_B \circ \iota_B)_n^* = (\mathrm{id}_B)_n^* = \mathrm{id}_{\pi_n(B,f(a_0))}$ and similarly $(r_A)_n^* \circ (\iota_A)_n^*$ is the identity). Moreover, since g_n^* is an isomorphism, we can consider the morphism $(r_A)_n^* \circ (g_n^*)^{-1} \circ (\iota_B)_n^*$. It is an inverse for f_n^*:

$$(r_A)_n^* \circ (g_n^*)^{-1} \circ (\iota_B)_n^* \circ f_n^* = (r_A)_n^* \circ (g_n^*)^{-1} \circ g_n^* \circ (\iota_A)_n^* = (r_A)_n^* \circ (\iota_A)_n^* = \mathrm{id}_{\pi_n(A,a_0)}$$

$$f_n^* \circ (r_A)_n^* \circ (g_n^*)^{-1} \circ (\iota_B)_n^* = (r_B)_n^* \circ g_n^* \circ (g_n^*)^{-1} \circ (\iota_B)_n^* = (r_B)_n^* \circ (\iota_B)_n^* = \mathrm{id}_{\pi_n(B,f(a_0))}$$

So f_n^* is an isomorphism. The same holds for f_0 by the same argument since π_0 is a functor too (this time, without basepoints). Hence f is a weak homotopy equivalence, as desired.

5.2.3 Axiom MC5: factorization

We will use factorizations of maps to prove axiom MC4 so we begin by proving MC5.

Factorization into a fibration and an acyclic cofibration. Let X and Y be topological spaces and $f : X \to Y$ be a continuous map. We would like to factorize f as $X \xrightarrow[\sim]{\iota} Z \overset{p}{\twoheadrightarrow} Y$ for some topological space Z. To ensure that p is a fibration, we will build it so that it has RLP with respect to all inclusions $\mathbb{D}^n \times \{0\} \xrightarrow{m_n} \mathbb{D}^n \times I$ (criterion of lemma (5.3)). The following argument is an example of *small object argument*.

We will build Z by "going a little further from X each time", keeping an "inclusion" of X into the spaces we build. Let $M = \{m_n : \mathbb{D}^n \times \{0\} \to \mathbb{D}^n \times I \mid n \in \mathbb{N}\}$. Since we are interested in finding lifts in diagrams of the form:

$$\begin{tikzcd}
\mathbb{D}^n \times \{0\} \arrow[r, "h"] \arrow[d, "m_n"'] & X \arrow[d, "f"] \\
\mathbb{D}^n \times I \arrow[r, "k"'] & Y
\end{tikzcd}$$

for all $n \in \mathbb{N}$, we consider $M_n^1 = \{(h,k) \in \mathbf{Top}(\mathbb{D}^n \times \{0\}, X) \times \mathbf{Top}(\mathbb{D}^n \times I, Y) \mid fh = km_n\}$ (equivalently, $M_n^1 = \mathbf{Top}^{\to}(m_n, f)$) the set of couples of maps defining a lifting problem as above.

Let $G^1(M,f)$ (we reproduce the construction presented in the article by Dwyer and Spaliński (1995)) be the pushout of the diagram (**Top** has all small colimits by subsection (5.2.1)):

$$\begin{tikzcd}
\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in M_n^1} \mathbb{D}^n \times \{0\} \arrow[r, "\sum_{n\in\mathbb{N}} \sum_{(g,h)\in M_n^1} g"] \arrow[d, "\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in M_n^1} m_n"'] & X \arrow[d, "\iota_1"] \arrow[ddr, bend left, "f"] & \\
\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in M_n^1} \mathbb{D}^n \times I \arrow[r, "j_1"'] \arrow[drr, bend right, "\sum_{n\in\mathbb{N}} \sum_{(g,h)\in M_n^1} h"'] \arrow[ur, phantom, "\ulcorner"] & G^1(M,f) \arrow[dr, dashed, "k_1"] & \\
& & Y
\end{tikzcd}$$

The map ι_1 obtained is a relative CW-inclusion and also a homotopy equivalence: by the universal property of the pushout, there is a map $\phi : G^1(M,f) \to X$ induced by id_X and the map $\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in M_n^1} \mathbb{D}^n \times I \longrightarrow X$ obtained by pre-composing $\sum_{n\in\mathbb{N}} \sum_{(g,h)\in M_n^1} g$ with the map that projects each cylinder $\mathbb{D}_n \times I$ onto its base $\mathbb{D}^n \times \{0\}$. By construction, $\phi\iota_1$ is the identity, and the other composition is homotopic to the identity of $G^1(M,f)$ (to obtain a homotopy, apply the functor $- \times I$ to the pushout. The result is still a pushout and the projection $X \times I \to X$, together with j_1 pre-composed with the disjoint union of homotopies projecting the cylinder onto their bases, induce a map $G^1(M, f) \times I \to G^1(M, f)$, which is the desired homotopy). Furthermore, ι_1 is the cobase change of a map having the LLP with respect to all Serre fibrations (by lemma (5.2) applied to the maps in M, that all have this property since $\mathbb{D}^n$ is a CW-complex for any $n \in \mathbb{N}$), so by proposition (2.10) ι_1 has the LLP with respect to all Serre fibrations. The map k_1 is induced by the universal property of the pushout.

We now repeat this process: define $G^2(M, f) = G^1(M, k_1)$, call $\iota_2 : G^1(M, f) \to G^2(M, f)$ and j_2 the obtained cobase change. The map ι_2 is again a relative CW-inclusion and a homotopy equivalence. By the universal property of the pushout, the maps $k_1 : G^1(M, f) \to Y$ and $\sum_{n\in\mathbb{N}} \sum_{(g,h)\in M_n^2} h : \coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in M_n^2} \mathbb{D}^n \times I \to Y$ induce a map $k_2 : G^2(M, f) \to Y$. Note that M_n^1 was replaced by $M_n^2 = \mathbf{Top}^{\to}(m_n, k_1)$. Inductively constructing $G^{n+1}(M, f) = G^1(M, k_n)$ for any $n \in \mathbb{N}$, we obtain maps $k_n : G^n(M, f) \to Y$ and CW-inclusions, homotopy equivalences $\iota_n : G^{n-1}(M, f) \to G^n(M, f)$ (define $G^0(M, f)$ as X) with LLP with respect to Serre fibrations, and maps $j_n : \coprod_{k\in\mathbb{N}} \coprod_{(g,h)\in M_k^n} \mathbb{D}^k \times I \to G^n(M, f)$. The situation is the following:

$$\begin{tikzcd}
 & & G^\infty(M,f) & & & \\
X \arrow[urr, dotted, "\iota"] \arrow[r, "\iota_1"] \arrow[drr, "f"'] & G^1(M,f) \arrow[ur, dotted, "\tilde{\iota}_1"] \arrow[r, "\iota_2"] \arrow[dr, "k_1"] & G^2(M,f) \arrow[u, dotted, "\tilde{\iota}_2"] \arrow[r, "\iota_3"] \arrow[d, "k_2"] & \dots \arrow[r] & G^n(M,f) \arrow[ull, dotted, "\tilde{\iota}_n"'] \arrow[r, "\iota_n"] \arrow[dll, "k_n"] & \dots \\
 & & Y & & &
\end{tikzcd}$$

where $G^\infty(M, f)$ is the sequential colimit of the row formed by the maps $\{\iota_n\}_{n\in\mathbb{N}^*}$. The universal property of the colimit provides us with a map $p : G^\infty(M, f) \to Y$, and $f = p \circ \iota$. If we show that ι is an acyclic cofibration and p is a fibration we are done. First of all we need the lemma:

Lemma 5.6 (Small object property)**.** *Let $h : A \to G^\infty(M, f)$ be a continuous map, with A a sequentially compact topological space. Then, there exists $k \in \mathbb{N}$ such that h factors through $G^k(M, f)$, i.e. $h = \tilde{\iota}_k \circ h'$ for some continuous map $h' : A \to G^k(M, f)$.*

Proof. This proof is inspired by the article by Hirschhorn (2019). We show that any sequentially compact subset of $G^\infty(M, f)$ intersects the interior of only finitely many cells we attached to X to form $G^\infty(M, f)$ ($(G^\infty(M, f), X)$ is a relative CW-complex). This will conclude the proof: the image of A by the continuous map h is sequentially compact, so it intersects the interiors of only finitely many cells $c_1, \dots, c_m$. Each of these cells c_i was attached at some step k_i. For $k := \max\{k_1, \dots, k_m\}$, h factors through $G^k(M, f)$.

For shorter notation let $G = G^\infty(M, f)$ and $G^k := G^k(M, f)$ $\forall k \in \mathbb{N}^*$, $G^0 = X$. Let Y be a sequentially compact subset of G. Let $\{c_i\}_{i\in I'}$ be the set of cells intersecting Y on their interior. For any $i \in I'$, choose a point b_i both in Y and in the interior of c_i (axiom of choice). The chosen points must be different for each cell (two cells intersect only if the boundary of one has been attached to the other, but in this case, the point is on the boundary and not the interior of the second cell). If $\{b_i\}_{i\in I'}$ does not accumulate anywhere in Y, we are done: since Y is sequentially compact, this can happen only if I' is finite.

Suppose for a contradiction that y is an accumulation point for $\{b_i\}_{i\in I'}$ in Y.We will show that there exists an open neighbourhood U of y in G that does not contain any point b_j for $j \in I'$, except maybe for y itself. Consider the smallest integer k such that $y \in G^k$. If $k = 0$ choose $U_k = X$. Otherwise, y belongs to the interior of some cell c_{i_k} for $i_k \in I'$. There is one point $b_{i_k} \in c_{i_k} \cap Y$. If $b_{i_k} = y$, U_k be the interior of c_{i_k}. It is an open neighbourhood of y in G^k, that does not contain any $b_j \neq y$ for $j \in I'$. Otherwise, just choose $U_k \setminus \{b_{i_k}\}$. We want to extend this subset to an open neighbourhood of y in G with the same property. We proceed by induction: for all $m \geq k$, we define an open subset U_m in G^m such that $b_j \in U_m \implies b_j = y$. Assume that $U_k, \dots U_m$ have already been defined. If U_m is not open in G^{m+1}, there must be some cell c with dimension n and an attaching map $a : \mathbb{S}^{n-1} \to G^m$ whose image intersects U_m. Because U_m is open in G^m, $a^{-1}(U_m)$ is open in $\mathbb{S}^{n-1}$.

The cell c comes from attaching a n-disk to G^m via a. Assume that c contains $b_j \neq y$ for some $j \in I$ in its interior. The open set $a^{-1}(U_m)$ can be enlarged into an open set in $\mathbb{D}^n$ that intersects $\mathbb{S}^{n-1}$ only on $a^{-1}(U_m)$, and does not contain the (preimage of the) point b_j (because $\{b_j\}$ is closed in $\mathbb{D}^n$). If there is no such b_j in c, we can choose any neighbourhood that intersects $\mathbb{S}^{n-1}$ only on $a^{-1}(U_m)$. Call N_c the corresponding subset of G^{m+1}. Let $U_{m+1} = U_m \cup \bigcup_{c \text{ a cell in } G^{m+1} \text{ intersecting } U_m} N_c$. Then U_{m+1} is an open set in G^{m+1} satisfying the desired condition (it is open because its intersection with G^m, namely U_m, is open in G^m and its intersection with any added cell in G^{m+1} is open by construction).

Now let $U = \bigcup_{m \geq k} U_m$. By construction $b_j \in U \iff b_j = y$ and U is an open neighbourhood of y in G since its intersection with X and any cell that was attached to form G from X is open by construction. But this is a contradiction: y cannot be an accumulation point of a sequence in $\{b_i\}_{i \in I'}$. Hence $\{b_i\}_{i \in I'}$ does not accumulate anywhere in Y: by sequential compactness, I' is finite and we are done. $\square$

We are now ready to show that p is a fibration. To do so, we check that the hypotheses of lemma (5.3) hold. Consider the lifting problem:

$$\begin{tikzcd}
\mathbb{D}^n \times \{0\} \arrow[r, "g"] \arrow[d, "m_n"'] & G^\infty(M,f) \arrow[d, "p"] \\
\mathbb{D}^n \times I \arrow[r, "h"'] & Y
\end{tikzcd}$$

Since $\mathbb{D}^n$ is sequentially compact, by lemma (5.6) the map g factors through $G^m(M,f)$ for some integer m. We have the following situation:

$$\begin{tikzcd}
\mathbb{D}^n \times \{0\} \arrow[r, "g"] \arrow[d, "m_n"'] & G^m(M,f) \arrow[r, "\iota_{m+1}"] \arrow[d, "k_m"'] & G^{m+1}(M,f) \arrow[r, "\tilde{\iota}_{m+1}"] \arrow[d, "k_{m+1}"'] & G^\infty(M,f) \arrow[d, "p"] \\
\mathbb{D}^n \times I \arrow[r, "h"'] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Since $(g,h) \in M_n^{m+1}$, we have a map:

$$\ell : \mathbb{D}^n \times I \xrightarrow{i} \coprod_{n \in \mathbb{N}} \coprod_{(h',h'') \in M_n^{m+1}} \mathbb{D}^n \times I \xrightarrow{j_{m+1}} G^{m+1}(M,f)$$

corresponding to the restriction of j_{m+1} to the copy of $\mathbb{D}^n \times I$ indexed by the pair (g,h). We claim that $\tilde{\iota}_{m+1} \circ \ell$ is the desired lift: indeed,

$$p \circ \tilde{\iota}_{m+1} \circ \ell = k_{m+1} \circ j_{m+1} \circ i = \left(\sum_{n \in \mathbb{N}} \sum_{(h',h'') \in M_n^{m+1}} h'' \right) \circ i = h$$

by construction and $\tilde{\iota}_{m+1} \circ \ell \circ m_n = \tilde{\iota}_{m+1} \circ j_{m+1} \circ i \circ m_n = \tilde{\iota}_{m+1} \circ \iota_{m+1} \circ g = g$.

Finally, p is a Serre fibration.

Moreover, ι is a cofibration: each map ι_n, $n \in \mathbb{N}$ has the LLP with respect to all Serre fibrations as we saw, and since lifting properties are stable by sequential colimits, ι has this same LLP. In particular, ι has the LLP with respect to acyclic Serre fibrations and is a cofibration.

Finally, ι is a weak homotopy equivalence: for $n \in \mathbb{N}$, consider the map induced on homotopy classes: $\iota_n^* : [\mathbb{S}^n, X]_* \to [\mathbb{S}^n, G^\infty(M,f)]_*$.

- This map is surjective: consider $g : \mathbb{S}^n \to G^\infty(M,f)$ a pointed map. Then g factors through some $G^k(M,f)$ ($\mathbb{S}^n$ is compact, we apply lemma (5.6)). Since ι_k is a homotopy equivalence, it is in particular a weak homotopy equivalence, and $(\iota_k)_n^* : [\mathbb{S}^n, X]_* \to [\mathbb{S}^n, G^k(M,f)]_*$ is bijective, so there exists $f : \mathbb{S}^n \to X$ such that $\iota_k \circ f$ is homotopic to g. Post-composing by the inclusion of $G^k(M,f)$ into $G^\infty(M,f)$, $[f]$ is also a preimage for $[g]$ by ι_n^*.

- This map is injective: suppose that $f, g : \mathbb{S}^n \to X$ are maps such that ιf is homotopic to ιg. We have to show that f was homotopic to g in the first place. Choose a homotopy $H : \mathbb{S}^n \times I \to G^\infty(M, f)$ between ιf and ιg. Again, the whole image of H lives in some $G^k(M, f)$ (once more by lemma (5.6), using that $\mathbb{S}^n \times I$ is compact). We get $(\iota_k)^*_n([f]) = (\iota_k)^*_n([g])$, and since ι_k is a weak equivalence, $[f] = [g]$ in $[\mathbb{S}^n, X]_*$ and we are done.

Therefore, ι is an acyclic cofibration. Then, we can set $\omega(f) = G^\infty(M, f)$, $\iota(f) = \iota$ and $\pi(f) = p$.

Remark 5.7. Here we proved that ι also has the LLP with respect to Serre fibrations that are not weak equivalences. We will re-use this fact later.

Factorization into an acyclic fibration and a cofibration We do the same proof, except that we replace M by the set of inclusions $M' = \{m'_n : \mathbb{S}^{n-1} \to \mathbb{D}^n \mid n \in \mathbb{N}\}$. Then, by lemma (5.4), the map p obtained will be an acyclic fibration. This time we cannot say that the maps ι_n, $n \in \mathbb{N}$ are homotopy equivalences (previously, we used the fact that $\mathbb{D}^n \times I \simeq \mathbb{D}^n \times \{0\}$, now we have $\mathbb{S}^{n-1} \not\simeq \mathbb{D}^n$), but they are still relative CW-inclusions. By lemma (5.5), they have the LLP with respect to acyclic Serre fibrations. This property is stable by sequential colimits by proposition (2.10), so the map ι has this same property: it is a cofibration. This time we can choose $\omega'(f) = G^\infty(M', f)$, $\iota'(f) = \iota$, and $\pi'(f) = p$.

5.2.4 Axiom MC4: lifting problems

The first part of axiom MC4 is just the definition of a cofibration: we defined them as having the LLP with respect to Serre fibrations that are also weak homotopy equivalences, i.e. acyclic fibrations.

For the second part: let $\iota : A \xhookrightarrow{\sim} B$ be a cofibration and a weak homotopy equivalence. We want to show that ι has the LLP with respect to all Serre fibrations. By what we showed in the previous subsection, we may write ι as a composition $A \xhookrightarrow[\sim]{\iota'} C \xrightarrow{p'} B$. By the 2 of 3 rule we already verified, p' is also a weak homotopy equivalence. Therefore, by definition of a cofibration, there exists a lift ℓ in the first diagram below, allowing us to express ι as a retract of ι' (second diagram):

$$
\begin{tikzcd}
A \arrow[r, hook, "\iota'", "\sim"'] \arrow[d, hook, "\iota"', "\wr"] & C \arrow[d, two heads, "p'", "\wr"'] \\
B \arrow[r, equal] \arrow[ur, dashed, "\ell"] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\iota"] & A \arrow[r, equal] \arrow[d, "\iota'"] & A \arrow[d, "\iota"] \\
B \arrow[r, dashed, "\ell"] \arrow[rr, bend right, equal] & C \arrow[r, dashed, "p'"] & B
\end{tikzcd}
$$

But from remark (5.7) in the proof of MC5, we know that ι' has the LLP with respect to Serre fibrations. This property is preserved by retracts (by lemma (2.10)), so ι has the LLP with respect to Serre fibrations.

This concludes the proof of theorem (5.1).

6 A model category structure for simplicial sets

In this section we introduce in details another example of a model category. We will define the category $\mathbf{sSet}$ of *simplicial sets*, which are rather combinatorial objects in nature, but can also be studied from a geometrical point of view by the use of the *geometric realization functor* from $\mathbf{sSet}$ to $\mathbf{Top}$. The model structure we study was defined by Quillen (1967).

6.1 What are simplicial sets?

Intuitively, simplicial sets represent constructions that can be made using basic "blocks" of different dimensions, namely points, segments, triangles, tetrahedron and so on. Each "block" is defined by a fixed number of vertices (for instance, a tetrahedron has four vertices) and has "faces" made of smaller blocks (faces of a segment are points for example), but can also be seen as a bigger, degenerate, block: a segment is just a flat triangle. Simplicial sets are however more complicated, for instance vertices do not uniquely define some kind of basic block, and the way in which blocks are glued together can be quite intricate. But this geometrical intuition is still useful to understand what is going on. In this section, we follow the book by Goerss and Jardine (1999).

Here is an abstract, categorical definition of a simplicial set.

Definition 6.1 (Simplicial set)**.** Let Δ be the category with objects the natural numbers, represented as ordered sets: $\mathrm{Ob}(\Delta) = \{[n] = \{0, 1, \dots, n\} \mid n \in \mathbb{N}\}$, and morphisms from $[n]$ to $[m]$ all order-preserving maps from $[n]$ to $[m]$. The *category of simplicial sets* is the category $_{\mathbf{S}}\mathbf{Set} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, the category of functors from Δ^{op} to the category of sets (equivalently, contravariant functors from Δ to $\mathbf{Set}$). For any *simplicial set* X (that is, an object of $_{\mathbf{S}}\mathbf{Set}$) and any $n \in \mathbb{N}$, we denote $X_n = X([n])$ the set of *n-simplices of X*. A *simplicial map* is a map between two simplicial sets (a natural transformation between them).

For $n, m \in N$, we will sometimes denote an order-preserving map $\sigma \in \Delta([n], [m])$ as a string $\sigma(0) \to \sigma(1) \to \dots \to \sigma(n)$.

The n-simplices in definition (6.1) represent the basic blocks in dimension n we talked about. A more rigorous definition of these basic blocks is:

Definition 6.2 (Standard simplices)**.** Let $n \in \mathbb{N}$. The *standard n-simplex* Δ^n is defined as the functor $\mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathbf{Set}$. The *topological standard n-simplex* is the topological space $|\Delta^n|$ with underlying set

$$\left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \middle| \sum_{i=0}^{n} x_i = 1, x_i \geq 0 \ \forall i \leq n \right\}$$

and the Euclidean subspace topology.

In particular, $|\Delta^0|$ is a point, $|\Delta^1|$ is a segment, $|\Delta^2|$ is a triangle, $|\Delta^3|$ is a tetrahedron and so on. There is a functor $R : \Delta \to \mathbf{Top}$ defined on objects by $R([n]) = |\Delta^n|$ and on morphisms by $\theta \in \Delta([n], [m]) \mapsto R(\theta) \in \mathbf{Top}(|\Delta^n|, |\Delta^m|)$ where $R(\theta)(x_0, \dots, x_n) = (y_0, \dots, y_m) \in |\Delta^m|$ with $y_i = \sum_{j \in \theta^{-1}(\{i\})} x_j$ for all $0 \leq i \leq m$ (by convention the empty sum has value 0).

The definition of the standard n-simplex is convenient in particular because it allows us to describe the n-simplices of a simplicial set X using simplicial maps $\Delta^n \to X$ via the Yoneda lemma (remark (6.10)).

The functor R is well-defined because given $\theta \in \Delta([n], [m])$ and $x = (x_0, \dots, x_n) \in |\Delta^n|$, we have $R(\theta)(x_0, \dots, x_n) = (y_0, \dots, y_m) \in |\Delta^m|$: indeed it has non-negative coordinates, summing up to one: $\sum_{i=0}^m y_i = \sum_{i=0}^m \sum_{j \in \theta^{-1}(\{i\})} x_j = \sum_{i \in \theta^{-1}([m])} x_i = \sum_{i=0}^n x_i = 1$. For the functoriality, note that $R(\mathrm{id}_{[n]})$ is the identity: the i-th coordinate of $R(\mathrm{id}_{[n]})(x_0, \dots, x_n)$ is $\sum_{j \in \mathrm{id}_{[n]}^{-1}(\{i\})} x_j = x_i$ for all $0 \leq i \leq n$, and if $\theta \in \Delta([n], [m])$, $\theta' \in \Delta([m], [k])$, the i-th coordinate of $R(\theta' \circ \theta)(x_0, \dots, x_n)$ is:

$$\sum_{j \in (\theta' \circ \theta)^{-1}(\{i\})} x_j = \sum_{\ell \in \theta'^{-1}(\{i\})} \sum_{j \in \theta^{-1}(\{\ell\})} x_j = \sum_{\ell \in \theta'^{-1}(\{i\})} (R(\theta)(x))_\ell = ((R(\theta') \circ R(\theta))(x))_i$$

For these standard simplices, we have a geometrical notion of "face" and "degeneracy": the boundary of a topological standard n-simplex is made of $n+1$ faces, which are themselves copies of the topological standard $(n-1)$-simplex. Conversely the topological standard $(n-1)$-simplex can be seen as a degenerate, "flat" version of the n-simplex, in the same way that a segment can be seen as a degenerate triangle. In the category Δ, this concept translates as follows:

Definition 6.3 (Faces and degeneracies)**.** For $[n]$ an object in Δ, the $n+1$ maps of the form:

$$\partial^i : [n-1] \to [n], \qquad 0 \to 1 \to 2 \to (i-1) \to (i+1) \to \dots \to n$$

for $0 \leq i \leq n$ are called *cofaces* (the symbol ∂ recalls a notion of boundary), and the $n+1$ maps of the form:

$$d^i : [n+1] \to [n], \qquad 0 \to 1 \to 2 \to \dots \to (i-1) \to i \to i \to (i+1) \to \dots \to n$$

for $0 \leq i \leq n$ are called *codegeneracies*. For X a simplicial set and $0 \leq i \leq n$, the maps

$$\partial_i = X((\partial^i)^{\mathrm{op}}) : X_n \to X_{n-1}$$

and

$$d_i = X((d^i)^{\mathrm{op}}) : X_n \to X_{n+1}$$

are called *faces* and *degeneracies* respectively.

A face map ∂_i can be thought of as sending any n-simplex of X to its i-th face. Conversely, the $(n+1)$-simplices of X contain degenerate copies of all m-simplices of X for $m \leq n$ via the degeneracy maps.

Remark 6.4. Actually, cofaces and codegeneracies generate by composition all morphisms in Δ. They satisfy some relations called the *cosimplicial identities*:

$$\begin{cases} \partial^j \partial^i = \partial^i \partial^{j-1} & \text{if } i < j \\ d^j \partial^i = \partial^i d^{j-1} & \text{if } i < j \\ d^j \partial^j = \mathrm{id} = d^j \partial^{j+1} & \\ d^j \partial^i = \partial^{i-1} d^j & \text{if } i > j+1 \\ d^j d^i = d^i d^{j+1} & \text{if } i \leq j \end{cases}$$

The proof that these identities hold is quite tedious, and goes by direct computation. For example we check that $d^j \partial^j = \mathrm{id}$. For $n \geq j$, the map $\partial^j : [n-1] \to [n]$ is represented by

$$0 \to 1 \to \cdots \to (j-1) \to \underbrace{j+1}_{\partial^j(j)} \to \cdots \to \underbrace{n}_{\partial^j(n-1)}$$

and $d^j : [n] \to [n-1]$ is represented by

$$0 \to 1 \to \cdots \to (j-1) \to \underbrace{j}_{d^j(j)} \to \underbrace{j}_{d^j(j+1)} \to \underbrace{j+1}_{d^j(j+2)} \to \cdots \to \underbrace{n-1}_{d^j(n)},$$

so their composition is represented by

$$0 \to 1 \to 2 \to (j-1) \to \underbrace{j}_{d^j(\partial^j(j))=d^j(j+1)} \to \underbrace{j+1}_{d^j(\partial^j(j+1))=d^j(j+2)} \to \cdots \to \underbrace{n-1}_{d^j(\partial^j(n-1))=d^j(n)},$$

namely the identity.

In particular, this means that a simplicial set is determined by the data of faces and degeneracies satisfying the opposite *simplicial identities* (in Δ^{op}, the order of composition is reversed).

May (1992) gives in his book (Chapter I, §2) an explicit decomposition of any map in Δ: consider $\theta \in \Delta([n],[m])$ for some $n, m \in \mathbb{N}$. Let $i_1 > i_2 > \cdots > i_s$ be the integers in $[m] \setminus \theta([n])$ (the integers that θ "jumps") and $j_1 < j_2 < \cdots < j_\ell$ the integers in $[n]$ such that $\theta(j_i) = \theta(j_i + 1)$ (the integers that θ "repeats" twice or more). Then:

$$\theta = \partial^{i_1} \cdots \partial^{i_s} d^{j_1} \cdots d^{j_\ell}$$

For example, consider the map $\theta \in \Delta([3],[5])$ represented by $0 \to 2 \to 2 \to 5$. Here we have $i_1 = 4, i_2 = 3, i_3 = 1$ and $j_1 = 1$. We check that $\partial^4 \partial^3 \partial^1 d^1 = \theta$: the poset $[3]$ is represented by $0 \to 1 \to 2 \to 3$. Applying d^1 we get $0 \to 1 \to 1 \to 2$, under ∂^1 we get $0 \to 2 \to 2 \to 3$, under ∂^3 we get $0 \to 2 \to 2 \to 4$ and finally applying ∂^4 we get $0 \to 2 \to 2 \to 5$, as desired.

Definition 6.5 (Non-degenerate)**.** Let X be a simplicial set. An n-simplex $x \in X_n$ is called *non-degenerate* if it is not contained in the image of any degeneracy.

Example 6.6. All the m-simplices of Δ^n for $m > n$ are degenerate: let $x \in \Delta([m],[n])$ be an m-simplex. Since $m > n$, x is not injective but order-preserving: there exists $j \in [m]$ with $x(j) = x(j+1)$. Then, the degeneracy part in the decomposition above is not empty, x is given by pre-composing some map y with a codegeneracy d. Then $x = \mathrm{Hom}_\Delta(d^{\mathrm{op}}, [n])(y)$ with $\mathrm{Hom}_\Delta(d^{\mathrm{op}}, [n])$ a degeneracy in Δ^n: x is degenerate.

6.2 The geometric realization and adjunction with Top

In this subsection, we define two adjoint functors between $_{\mathbf{s}}\mathbf{Set}$ and **Top**. One of them is used in the definition of the Quillen model structure on $_{\mathbf{s}}\mathbf{Set}$.

Definition 6.7 (Singular set)**.** For T a topological space, let $S(T) : \Delta^{\mathrm{op}} \to \mathbf{Set}$ be the simplicial set defined on objects by $[n] \mapsto \mathbf{Top}(|\Delta^n|, T)$ and on morphisms by $f^{\mathrm{op}} \in \Delta^{\mathrm{op}}([n],[m]) \mapsto S(T)(f^{\mathrm{op}})$ where $S(T)(f^{\mathrm{op}})(g) = g \circ R(f)$ for any $g \in \mathbf{Top}(|\Delta^n|, T)$. To a continuous map $h \in \mathbf{Top}(T, T')$, we associate the simplicial map $S(h) : S(T) \to S(T')$ with $S(h)_n : \mathbf{Top}(|\Delta^n|, T) \to \mathbf{Top}(|\Delta^n|, T')$ given by post-composition by h for all $n \in \mathbb{N}$. Then $S : \mathbf{Top} \to {}_{\mathbf{S}}\mathbf{Set}$ is a functor, and $S(T)$ is called the *singular set associated to T*.

We check the functoriality of S and that is it well-defined:

- For any topological space T, we have to check that $S(T)$ is a simplicial set, i.e. a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. For all $n \in \mathbb{N}$, $S(T)\left(\mathrm{id}_{[n]}^{\Delta^{\mathrm{op}}}\right) = S(T)\left(\left(\mathrm{id}_{[n]}^{\Delta}\right)^{\mathrm{op}}\right)$ is the identity: it sends a map g to $g \circ R(\mathrm{id}_{[n]}^{\Delta}) = g \circ \mathrm{id}_{|\Delta^n|} = g$. Given maps $\theta^{\mathrm{op}} \in \Delta^{\mathrm{op}}([n],[m])$ and $\upsilon^{\mathrm{op}} \in \Delta^{\mathrm{op}}([m],[k])$, we have for any $g \in \mathbf{Top}(|\Delta^n|, T)$:

$$\begin{aligned} S(T)(\upsilon^{\mathrm{op}}\theta^{\mathrm{op}})(g) &= S(T)((\theta\upsilon)^{\mathrm{op}})(g) = g \circ R(\theta\upsilon) = g \circ R(\theta) \circ R(\upsilon) \\ &= S(T)(\upsilon^{\mathrm{op}})(g \circ R(\theta)) = (S(T)(\upsilon^{\mathrm{op}}) \circ S(T)(\theta^{\mathrm{op}}))(g) \end{aligned}$$

  hence the functoriality of $S(T)$.

- For any $g \in \mathbf{Top}(T, T')$, $S(g) : S(T) \to S(T')$ is a natural transformation. Indeed, for all $n, m \in \mathbb{N}$ and $\theta \in \Delta([m],[n])$, there is a commutative diagram:

$$\begin{tikzcd} S(T)_n \arrow[r, "S(g)_n"] \arrow[d, "S(T)(\theta^{\mathrm{op}})"'] & S(T')_n \arrow[d, "S(T')(\theta^{\mathrm{op}})"] \\ S(T)_m \arrow[r, "S(g)_m"'] & S(T')_m \end{tikzcd}$$

  since for all $h \in S(T)_n = \mathbf{Top}(|\Delta^n|, T)$, we have:

$$S(T')(\theta^{\mathrm{op}})(S(g)_n(h)) = S(T')(\theta^{\mathrm{op}})(g \circ h) = g \circ h \circ R(\theta) = S(g)_m(h \circ R(\theta)) = S(g)_m(S(T)(\theta^{\mathrm{op}})(h)).$$

- For the functoriality of S, the map $S(\mathrm{id}_T)_n$ for $n \in \mathbb{N}$ is obtained by post-composition with id_T: it is the identity. If $g \in \mathbf{Top}(T, T')$ and $g' \in \mathbf{Top}(T', T'')$, for any $n \in \mathbb{N}$ and $h \in S(T)_n$, we have $S(g'g)_n(h) = (g'g) \circ h = g' \circ (gh) = S(g')_n(S(g)_n(h))$ hence $S(g' \circ g) = S(g') \circ S(g)$.

Therefore, S is a (well-defined) functor from $\mathbf{Top}$ to ${}_{\mathbf{S}}\mathbf{Set}$.

In the other direction, we give two different but equivalent definitions for the *geometric realization* of a simplicial set. The first one is taken from the book “Simplicial objects in algebraic topology” by May (1992) (Chapter III), and illustrates well the geometric intuition behind simplicial sets, whereas the second one, which can be found in the book by Goerss and Jardine (1999) (Paragraph I.2.), is a categorical description, more convenient for some proofs we will do later.

Definition 6.8 (Geometric realization I)**.** For X a simplicial set, the *geometric realization of X* is the topological space:

$$|X| = \left(\coprod_{n \in \mathbb{N}} (X_n \times |\Delta^n|)\right) \big/ \sim$$

with $\sim$ the equivalence relation generated by the identifications

$$\begin{aligned} (d_i(x_n), (u_0, \ldots, u_{n+1})) &\sim (x_n, (u_0, \ldots, u_{i-1}, u_i + u_{i+1}, u_{i+2}, \ldots, u_{n+1})) \\ (\partial_i(x_n), (v_0, \ldots, v_{n-1})) &\sim (x_n, (v_0, \ldots, v_{i-1}, 0, v_i, v_{i+1}, \ldots, v_{n-1})) \end{aligned}$$

for all $x_n \in X_n$, $(u_0, \ldots, u_{n+1}) \in |\Delta^{n+1}|$ and $(v_0, \ldots, v_{n-1}) \in |\Delta^{n-1}|$, with the discrete, Euclidean, and quotient topologies for X_n, $|\Delta^n|$, and $|X|$ respectively.

This defines a functor $|\cdot| : {}_{\mathbf{S}}\mathbf{Set} \to \mathbf{Top}$ sending a simplicial map $f \in {}_{\mathbf{S}}\mathbf{Set}(X, Y)$ to $|f| : |X| \to |Y|$ with $|f|([x_n, u]) = [f_n(x_n), u]$ for all $n \in \mathbb{N}$, $x_n \in X_n$ and $u \in |\Delta^n|$.

Definition 6.9 (Geometric realization II)**.** For X a simplicial set, the *simplex category for* X, written $\Delta{\downarrow}X$, has objects all simplicial maps $\Delta^n \to X$ and arrows between $\sigma : \Delta^n \to X$ and $\tau : \Delta^m \to X$ the simplicial maps $\theta : \Delta^n \to \Delta^m$ such that:

$$\begin{tikzcd}
\Delta^n \arrow[d, "\theta"'] \arrow[dr, "\sigma"] & \\
\Delta^m \arrow[r, "\tau"'] & X
\end{tikzcd}$$

is commutative. Let $F : \Delta{\downarrow}X \to \mathbf{Top}$ be the functor remembering only the source Δ^n of an object $\Delta^n \to X$ in $\Delta{\downarrow}X$ and sending it to $|\Delta^n|$. The *geometric realization of* X is the simplicial set $|X| = \mathrm{colim}_{\Delta\downarrow X} F$.

Remark 6.10. Thanks to the Yoneda lemma, we can give a reformulation of the second definition: a simplicial set X is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. In view of the definition of Δ^n, the Yoneda lemma states that there is a bijection between $\mathrm{Nat}(\Delta^n, X) = \mathbf{_sSet}(\Delta^n, X)$ and $X([n]) = X_n$, natural in $[n]$ and X (cf example 1.7 in the book by Goerss and Jardine (1999)). Hence the simplex category $\Delta{\downarrow}X$ for X can be seen as the category $E(X)$ with $\mathrm{Ob}(E(X)) = \bigsqcup_{n\in\mathbb{N}} X_n$ and for any $x_n \in X_n$, $x_m \in X_m$, $E(X)(x_n, x_m) = \{\theta \in \Delta([n],[m]) \mid X(\theta^{\mathrm{op}})(x_m) = x_n\}$. Then $|X| = \mathrm{colim}_{E(X)} \tilde{F}$, where $\tilde{F} : E(X) \to \mathbf{Top}$ sends an n-simplex x_n to $|\Delta^n|$, and $\theta \in E(X)(x_n, x_m)$ to $R(\theta) \in \mathbf{Top}(|\Delta^n|, |\Delta^m|)$.

Remark 6.11. The Yoneda lemma also implies that $X = \mathrm{colim}_{\Delta\downarrow X} F'$ where $F' : \Delta{\downarrow}X \to \mathbf{_sSet}$ only remembers the source of an object in $\Delta{\downarrow}X$. Indeed, for any object $\sigma : \Delta^n \to X$ in $\Delta{\downarrow}X$, σ itself is a simplicial map $\Delta^n \to X$. Given Y another simplicial set, any collection of maps $y_\sigma : \Delta^n \to Y$ (for any object $\sigma : \Delta^n \to X$ in $\Delta{\downarrow}X$) making the suitable diagram commute, induces a simplicial map $f : X \to Y$: let $f_n : X_n \to Y_n$ given by $X_n \cong \mathbf{_sSet}(\Delta^n, X) \to \mathbf{_sSet}(\Delta^n, Y) \cong Y_n$ where the map in the middle sends σ to y_σ, and the two bijections on the sides are the ones from the Yoneda lemma. Hence X satisfies the universal property of the colimit for F', as desired. Equivalently, $X = \mathrm{colim}_{E(X)} \tilde{F}'$.

Example 6.12. The geometric realization of the standard n-simplex Δ^n is the standard topological n-simplex $|\Delta^n|$, justifying the notation. We want to show that this standard simplex is $\mathrm{colim}_{\Delta\downarrow\Delta^n} F'$. The identity id_{Δ^n} is terminal in $\Delta{\downarrow}\Delta^n$: for any $\sigma : \Delta^m \to \Delta^n$ there is a diagram:

$$\begin{tikzcd}
\Delta^m \arrow[d, "\sigma"'] \arrow[dr, "\sigma"] & \\
\Delta^n \arrow[r, equal] & \Delta^n
\end{tikzcd}$$

so if X is a topological space with maps $|\Delta^m| \to X$ for all objects $\Delta^m \to \Delta^n$ in $\Delta{\downarrow}\Delta^n$, there is in particular a map $|\Delta^n| \to X$ corresponding to id_{Δ^n}, that makes the suitable diagram commute because of the fact that id_{Δ^n} is terminal in $\Delta{\downarrow}\Delta^n$. Hence the standard topological n-simplex is the geometric realization of the standard n-simplex.

Lemma 6.13 (Equivalence of the definitions)**.** *For any simplicial set X, the two geometric realizations of definitions (6.8) and (6.9) are isomorphic in* $\mathbf{Top}$ *(homeomorphic as topological spaces).*

Proof. By the direct construction of colimits in subsection (5.2.1), $\mathrm{colim}_{E(X)} \tilde{F}$ is given by the quotient $U/_{\approx}$ where $U = \bigsqcup_{d\in\mathrm{Ob}(E(x))} \tilde{F}(d) = \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{x_n\in X_n} \tilde{F}(x_n) = \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{x_n\in X_n} |\Delta^n|$ and $\approx$ is generated by $(d, x) \approx (d', \tilde{F}(f)(x))$ for all $f \in E(x)(d, d')$. Since X_n has the discrete topology, U can also be written as $\bigsqcup_{n\in\mathbb{N}}(X_n \times |\Delta^n|)$. Since the cofaces and codegeneracies generate the maps in Δ (remark (6.4)), the relation $\approx$ is generated by the identifications $(d, x) \approx (d', \tilde{F}(f)(x))$ for f a codegeneracy or a coface. Let $x = (x_0, \ldots, x_n) \in |\Delta^n|$. When $f = d^i \in \Delta([n], [n-1])$ is a codegeneracy, we have $\tilde{F}(f)(x) = R(f)(x) = (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n)$ (because $(d^i)^{-1}(i) = \{i, i+1\}$). When $f = \partial^i \in \Delta([n], [n+1])$ is a coface, $\tilde{F}(f)(x) = R(f)(x) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$ (because $(\partial^i)^{-1}(\{i\}) = \emptyset$). These are exactly the identifications described for the relation $\sim$ in definition (6.8). Hence the relations $\sim$ and $\approx$ coincide. $\square$

Here is the advertised adjunction:

Proposition 6.14 (Adjunction between $\mathbf{_sSet}$ and $\mathbf{Top}$)**.** *The geometric realization functor is left-adjoint to the singular set functor:*

$$|\cdot| : \mathbf{_sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : S$$

Proof. We have to establish a natural bijection $\mathbf{Top}(|X|, Y) \cong {}_{\mathbf{S}}\mathbf{Set}(X, SY)$ for any simplicial set X and topological space Y. Using definition (6.9), remark (6.10) and their notation:

$$\begin{aligned}\mathbf{Top}(|X|, Y) &= \mathbf{Top}(\operatorname{colim}_{E(X)} \tilde{F}, Y) \\ &\cong \lim_{E(X)} \operatorname{Hom}_{\mathbf{Top}}(\tilde{F}(-), Y) \\ &\cong \lim_{E(X)} \operatorname{Hom}_{{}_{\mathbf{S}}\mathbf{Set}}(\tilde{F}'(-), SY) \\ &\cong {}_{\mathbf{S}}\mathbf{Set}(\operatorname{colim}_{E(X)} \tilde{F}', SY) \\ &\cong {}_{\mathbf{S}}\mathbf{Set}(X, SY)\end{aligned}$$

- For the first bijection, we want to show that $\mathbf{Top}(\operatorname{colim}_{E(X)} \tilde{F}, Y)$ satisfies the universal property of the limit for the functor $\operatorname{Hom}_{\mathbf{Top}}(\tilde{F}(-), Y) : E(X)^{\mathrm{op}} \to \mathbf{Set}$. Here $\mathbf{Top}$ can be replaced by any locally small category (so that the Hom functor takes values in $\mathbf{Set}$), the argument is not specific to our setting. For each object d of $E(X)$ there is $n \in \mathbb{N}$ such that $d \in X_n$. There is a map $\mathbf{Top}(\operatorname{colim}_{E(X)} \tilde{F}, Y) \to \mathbf{Top}(\tilde{F}(d), Y) = \mathbf{Top}(|\Delta^n|, Y)$ given by precomposition by the natural map from $\tilde{F}(d)$ to $\operatorname{colim}_{E(X)} \tilde{F}$. Moreover, consider any set S with maps $r_d : S \to \mathbf{Top}(\tilde{F}(d), Y)$ for any object d in $E(X)$, making the suitable diagram commute. In this situation, for any $s \in S$, by the universal property of the colimit, the maps $r_d(s) : \tilde{F}(d) \to Y$ induce a continuous map $\operatorname{colim}_{E(X)} \tilde{F} \to Y$. This defines a map $S \to \mathbf{Top}(\operatorname{colim}_{E(X)} \tilde{F}, Y)$, hence the set $\mathbf{Top}(\operatorname{colim}_{E(X)} \tilde{F}, Y)$ satisfies the universal property of the limit and is (in bijection with) $\lim_{E(X)} \operatorname{Hom}_{\mathbf{Top}}(\tilde{F}(-), Y)$. By the same argument, $\lim_{E(X)} \operatorname{Hom}_{{}_{\mathbf{S}}\mathbf{Set}}(\tilde{F}'(-), SY) \cong {}_{\mathbf{S}}\mathbf{Set}(\operatorname{colim}_{E(X)} \tilde{F}', SY)$.
- For the second bijection, let $x_n \in X_n$ be an n-simplex of X. Then, by definition of the singular set, $\mathbf{Top}(\tilde{F}(x_n), Y) = \mathbf{Top}(|\Delta^n|, Y) = (SY)_n$, but by the Yoneda lemma there is a natural bijection (remark (6.10)) $SY_n = SY([n]) \cong {}_{\mathbf{S}}\mathbf{Set}(\Delta^n, SY) = {}_{\mathbf{S}}\mathbf{Set}(\tilde{F}'(x_n), SY)$.
- The last identification follows from remark (6.11).

□

6.3 The Quillen model structure on ${}_{\mathbf{S}}$Set and first properties

Theorem 6.15 (The Quillen model structure on ${}_{\mathbf{S}}\mathbf{Set}$). *In the category* ${}_{\mathbf{S}}\mathbf{Set}$ *of simplicial sets, define:*

- $\mathbf{We}({}_{\mathbf{S}}\mathbf{Set})$ *as the class of simplicial maps whose geometric realization is a weak homotopy equivalence.*
- $\mathbf{Cofib}({}_{\mathbf{S}}\mathbf{Set})$ *as the class of simplicial maps that are injective on each set of n-simplices, i.e. a map $f \in {}_{\mathbf{S}}\mathbf{Set}(X, Y)$ is a cofibration if and only if $f_n \in \mathbf{Set}(X_n, Y_n)$ is injective for all $n \in \mathbb{N}$.*
- $\mathbf{Fib}({}_{\mathbf{S}}\mathbf{Set})$ *as the class of simplicial maps having the RLP with respect to all maps that are both cofibrations and weak equivalences.*

This choice endows ${}_{\mathbf{S}}\mathbf{Set}$ *with a model category structure, called the* Quillen model structure *(see the book by Quillen (1967)).*

To begin with, notice that these classes of maps are stable by composition and contain all identities, and that the 2 of 3 rule holds:

- Consider simplicial maps $f : X \to Y$, $g : Y \to Z$ and $gf : X \to Z$ and suppose that two of them are weak equivalences. By functoriality of the geometric realization $|gf| = |g| \circ |f|$, so two of the three maps $|f|$,$|g|$ and $|g| \circ |f|$ are weak homotopy equivalences. By the 2 of 3 rule for the Quillen model structure on $\mathbf{Top}$ (section (5)), all of them are weak homotopy equivalences. Thus, f, g and gf are weak equivalences, and the 2 of 3 rule holds in ${}_{\mathbf{S}}\mathbf{Set}$ as well. By functoriality again, the identities in ${}_{\mathbf{S}}\mathbf{Set}$ induce the identities when passing to the geometric realization, that is, weak homotopy equivalences: identities in ${}_{\mathbf{S}}\mathbf{Set}$ are weak equivalences as well.

- Composition in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ is defined pointwise: if $f : X \to Y$ and $g : Y \to Z$ are simplicial maps, $g \circ f$ is the natural transformation defined by $(g \circ f)_{[n]} = g_{[n]} \circ f_{[n]}$ for all $[n] \in \Delta^{\mathrm{op}}$. Hence if f and g are cofibrations, each $(g \circ f)_{[n]}$ for $n \in \mathbb{N}$ is injective as a composition of injective maps, i.e. $g \circ f$ is a cofibration. For all $n \in \mathbb{N}$, the identity of a simplicial set X induces the identity on X_n, which is injective, i.e. id_X is a cofibration.

- Once the class of cofibrations is defined, the class of fibrations is defined by a RLP so by the dual of proposition (2.10), the class of fibrations is stable by composition and contains the identities.

Axiom MC1: limits and colimits

The category **Set** has all small limits and colimits: the construction is the same as in **Top**, forgetting about the topologies. Indeed, the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is both a left and right adjoint, for the functors endowing a set with the indiscrete, respectively discrete topology, so it preserves colimits, respectively limits. Using this fact, we prove that $_{\mathbf{S}}\mathbf{Set}$, as a category of functors between a small category (namely Δ^{op}) and a category with all small limits and colimits, has itself all small limits and colimits. They can be computed pointwise in the following sense: for D a small category and $F : D \to {_{\mathbf{S}}\mathbf{Set}}$ a functor, if F_n denotes the functor $D \to \mathbf{Set}$, $d \mapsto F(d)_n$, for all $n \in \mathbb{N}$, then:

$$(\lim_D F)_n = \lim_D F_n, \qquad\qquad (\operatorname{colim}_D F)_n = \operatorname{colim}_D F_n.$$

We only prove the case of colimits, the proof for limits being symmetric. Given a category D and a functor F as above, we define a simplicial set X with $X_n = \operatorname{colim}_D F_n$. For all $\theta \in \Delta([n],[m])$, the maps $F(d)(\theta^{\mathrm{op}}) : F(d)_m \to F(d)_n$ followed by the natural "inclusions" $F(d)_n \to \operatorname{colim}_D F_n$ for all $d \in \mathrm{Ob}(D)$ induce a map $X(\theta^{\mathrm{op}}) : X_m = \operatorname{colim}_D F_m \to X_n = \operatorname{colim}_D F_n$ by the universal property of colimits. The functoriality comes from the fact that the image by X of a map is induced "pointwise" by the image of this map under the functors $\{F(d)\}_{d \in \mathrm{Ob}(D)}$, and their functoriality passes to the colimit. There is a natural transformation $F(d) \to X$ for all $d \in \mathrm{Ob}(D)$, with $F(d)_n \to X_n = \operatorname{colim}_D F_n$ the natural "inclusion".

Given another simplicial set Y with maps $F(d) \to Y$ for all $d \in \mathrm{Ob}(D)$ making the suitable diagrams commute, there is an induced map $X \to Y$ defined pointwise by the universal property of the colimit, since $X_n = \operatorname{colim}_D F_n$ for all $n \in \mathbb{N}$. Therefore $X = \operatorname{colim}_D F$ is the desired colimit.

6.4 Useful lemmas

To prove that the other axioms hold, we need a few lemmas. This subsection is inspired by lecture notes by Jardine (2018). First, a definition:

Definition 6.16 (Boundary of Δ^n)**.** The simplicial set $\partial\Delta^n$ is defined as the smallest "sub-simplicial set" in Δ^n containing all its simplices of order strictly less than n. More precisely, $(\partial\Delta^n)_j = (\Delta^n)_j$ for all $j < n$ and if $j \geq n$, $(\partial\Delta^n)_j$ contains only iterated degeneracies of the simplices of lower order.

Lemma 6.17 (Acyclic fibration criterion)**.** *Let p be a simplicial map having the RLP with respect to all inclusions $\{b_n : \partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\}$. Then p is an acyclic fibration.*

Proof. We begin by proving that p has the RLP with respect to all cofibrations. In particular, it is a fibration. Let $\iota : A \hookrightarrow B$ be a cofibration. This means that each ι_n, $n \in \mathbb{N}$, is injective, in particular, ι can be seen as an inclusion of A in B. We would like to obtain the inclusion $A \hookrightarrow B$ as a sequential colimit. By analogy to the n-skeleton of a CW-complex:

Definition 6.18 (n-skeleton)**.** Let $n \in \mathbb{N}$ and B be a simplicial set. The *n-skeleton* of B, written $\mathrm{sk}_n B$, is the sub-simplicial set of B consisting of all m-simplices of B for $m \leq n$ and all their degeneracies.

Similarly to topological spaces, we have a diagram:

$$A \hookrightarrow A \cup \mathrm{sk}_0 B \hookrightarrow A \cup \mathrm{sk}_1 B \hookrightarrow \dots$$

with colimit B (simply a union here, because the colimit is computed pointwise on the level of sets by subsection (6.3)) and first natural map the inclusion $\iota : A \to B$. Setting $\mathrm{sk}_{-1} B = \emptyset$, for all $i \geq 0$

the inclusion $A \cup \mathrm{sk}_{i-1}B \to A \cup \mathrm{sk}_i B$ is given by the pushout:

$$\begin{tikzcd}
\coprod_{x \in B_i \setminus A_i} \partial\Delta^i \arrow[rr, "\sum_{x\in B_i\setminus A_i} \gamma_i(x)|_{\partial\Delta^i}"] \arrow[d, "\coprod_{x\in B_i\setminus A_i} b_i"'] & & A \cup \mathrm{sk}_{i-1}B \arrow[d] \\
\coprod_{x \in B_i \setminus A_i} \Delta^i \arrow[rr, "\sum_{x\in B_i\setminus A_i} \gamma_i(x)"'] & & A \cup \mathrm{sk}_i B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where $\gamma_i : B_i \to \mathbf{sSet}(\Delta^i, B)$ is the Yoneda bijection (see remark (6.10)). Indeed, on the set of k-simplices for $k \le i-1$, the two vertical arrows are just identities and the horizontal arrows coincide. For $k > i$, all k-simplices in the diagram above except the ones in A are degenerate (see example (6.6)). For i-simplices, the sets of simplices on the upper row are the subsets of degenerate simplices for the sets on the lower row (except for A_i). Only degenerate simplices are identified in the pushout. In Δ^i_i, only $\mathrm{id}_{[i]}$ is non-degenerate, and by the definition of the bijection in the Yoneda lemma, $\gamma_i(x)_{[i]}$ sends it to x. So by taking the disjoint union on all $x \in B_i \setminus A_i$, we finally get all the simplices of $A \cup \mathrm{sk}_i B$: the diagram above is (pointwise, on the sets of simplices) a pushout.

In $\mathbf{sSet}$, lemma (5.2) (lifting properties for maps obtained by disjoint union) holds as well, because colimits are computed pointwise at the level of sets of simplices. For all $i \in \mathbb{N}$, since b_i has the LLP with respect to p by hypothesis, the disjoint union on the left-hand side of the diagram too, and by cobase change, the inclusion $A \cup \mathrm{sk}_{i-1}B \to A \cup \mathrm{sk}_i B$ has the LLP with respect to p as well. Using the property (2.10), the inclusion $\iota : A \to B$, which is the first natural map in the colimit, has the LLP with respect to p, equivalently p has the RLP with respect to ι.

Then, we prove that $p : X \to Y$ is a weak equivalence. Since the map $\emptyset \to J$ is injective for any set J, any simplicial set is cofibrant (the initial object is the empty simplicial set). There is also a cofibration $i : X \amalg X \hookrightarrow X \times \Delta^1$. Indeed, $(X \amalg X)_n = X_n \sqcup X_n$ and $(X \times \Delta^1)_n = X_n \times \Delta([n],[1])$. There are injections $X_n \sqcup X_n \to X_n \times \Delta([n],[1])$ sending the first copy of X_n to $X_n \times \{\underline{0}\}$ and the second one to $X_n \times \{\underline{1}\}$, where $\underline{0}$ and $\underline{1}$ denote the constant maps 0 and 1 respectively. Therefore, lifts exist in the following diagrams:

$$\begin{tikzcd}
\emptyset \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "p"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "\exists s"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \amalg X \arrow[r, "sp+\mathrm{id}_X"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\
X \times \Delta^1 \arrow[r, "p\circ\pi_1"'] \arrow[ur, dashed, "\exists h"] & Y
\end{tikzcd}$$

(the second diagram is commutative because $ps = \mathrm{id}_Y$) with $\pi_1 : X \times \Delta^1 \to X$ the first natural projection. The geometric realization functor, as a left adjoint (cf proposition (6.14)), preserves colimits, and gives a diagram:

$$\begin{tikzcd}
|X| \amalg |X| \arrow[r, "|s||p|+\mathrm{id}_{|X|}"] \arrow[d, "|i|"'] & |X| \arrow[d, "|p|"] \\
|X \times \Delta^1| \arrow[r, "|p||\pi_1|"'] \arrow[ur, "|h|"] & Y
\end{tikzcd}$$

If we show that $|X \times \Delta^1| \approx |X| \times I$ and that $|i|$ corresponds to the usual inclusion $|X| \amalg |X| \to |X| \times I$, $|h|$ is a homotopy between $|s| \circ |p|$ and the identity. Since $|p| \circ |s| = |ps| = |\mathrm{id}_Y| = \mathrm{id}_{|Y|}$, this implies that $|p|$ is a homotopy equivalence, in particular it is a weak equivalence in **Top**, so p is a weak equivalence in $\mathbf{sSet}$.

- First, we check that taking the product with Δ^1 preserves colimits in $\mathbf{sSet}$. Consider D a small category, and $G : D \to \mathbf{sSet}$. We want to show that $(\mathrm{colim}_D\, G) \times \Delta^1 \cong \mathrm{colim}_D\,(G \times \Delta^1)$ where $G \times \Delta^1 : D \to \mathbf{sSet}$ sends an object d in D to $G(d) \times \Delta^1$ (and sends a map f to $G(f) \times \mathrm{id}_{\Delta^1}$). We define a bijection on the level of the sets of simplices: for all $n \in \mathbb{N}$, we have

$$\left((\mathrm{colim}_D\, G) \times \Delta^1\right)_n = (\mathrm{colim}_D\, G)_n \times (\Delta^1)_n = (\mathrm{colim}_D\, G_n) \times \Delta^1_n$$
$$\left(\mathrm{colim}_D\,(G \times \Delta^1)\right)_n = \mathrm{colim}_D\,(G \times \Delta^1)_n = \mathrm{colim}_D\,(G_n \times \Delta^1_n).$$

These two sets are in bijection, because the functor $\mathbf{Set} \to \mathbf{Set}$ obtained by taking the product with a fixed set S is a left adjoint (with right adjoint $\mathrm{Hom}_{\mathbf{Set}}(S,-)$). For faces and degeneracies everything happens in the same way at the level of sets. Hence the desired isomorphism in $\mathbf{sSet}$.

- By definition (6.9) of the geometric realization with colimits and remark (6.11):

$$\begin{aligned}
|X\times\Delta^1| &= \left|(\operatorname{colim}_{\Delta\downarrow X}F')\times\Delta^1\right| \\
&\approx \left|\operatorname{colim}_{\Delta\downarrow X}(F'\times\Delta^1)\right| \\
&\approx \operatorname{colim}_{\Delta\downarrow X}\left|F'\times\Delta^1\right| && \text{because } |\cdot| \text{ is a left adjoint} \\
&\overset{(\star)}{\approx} \operatorname{colim}_{\Delta\downarrow X}(|F'|\times|\Delta^1|) && \text{since all images by } F' \text{ of objects in } \Delta\downarrow X \text{ are standard simplices} \\
&\approx \operatorname{colim}_{\Delta\downarrow X}(F\times I) && \text{since } I\approx|\Delta^1| \\
&\approx (\operatorname{colim}_{\Delta\downarrow X}F)\times I && \text{since } -\times I:\mathbf{Top}\to\mathbf{Top} \text{ is a left adjoint} \\
&= |X|\times I\approx|X|\times|\Delta^1|
\end{aligned}$$

  The step $(\star)$ is proved below.

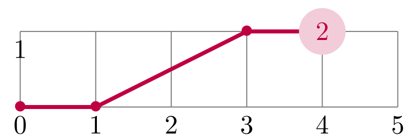
- The inclusion $i : X\amalg X\to X\times\Delta^1$ remains the usual inclusion $|X|\amalg|X|\to|X|\times I$ when passing to the geometric realization. The only step that may cause problems in the computation above is $(\star)$. But we show while proving that $|\Delta^p\times\Delta^1|\approx|\Delta^p|\times|\Delta^1|$ that the inclusion is preserved, for all $p\in\mathbb{N}$. Hence $|i|$ is the usual inclusion of $|X|\amalg|X|$ into its cylinder.

We now prove the step $(\star)$. By the universal property of products, the geometric realizations of the canonical projections, $|\Delta^p\times\Delta^1|\xrightarrow{|\pi_0|}|\Delta^p|$ and $|\Delta^p\times\Delta^1|\xrightarrow{|\pi_1|}|\Delta^1|$, induce a (continuous) map $f:|\Delta^p\times\Delta^1|\to|\Delta^p|\times|\Delta^1|$. In order to find an inverse, we would like to explicit f.

For this we need to understand the simplices of $\Delta^p\times\Delta^1$. Let $n\in\mathbb{N}$. By definition,

$$(\Delta^p\times\Delta^1)_n=\Delta^p_n\times\Delta^1_n=\Delta([n],[p])\times\Delta([n],[1]).$$

For an n-simplex, if we look at each of the two maps composing it as defining one of the two coordinates of a point in the plane, choosing such a simplex amounts to choosing a path in $[0,p]\times[0,1]$ with integer "nodes". At each step this path can only move up and/or right (or not move at all). If $n=4$ and $p=5$, an example is the path:

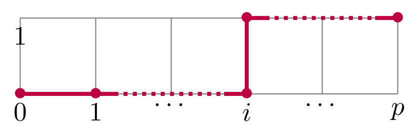


(the node marked "2" means that the path stays at this point for two consecutive steps) corresponding to the element $(0\to1\to3\to4\to4, 0\to0\to1\to1\to1)$.

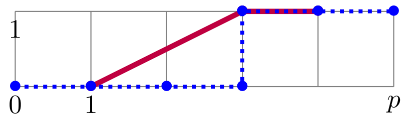
Suppose that $(\theta,\theta')\in(\Delta^p\times\Delta^1)_n$ is non degenerate. The path representing it cannot stay twice at the same node: this would imply that there exists $0\le j\le n-1$ with $\theta(j)=\theta(j+1)$ and $\theta'(j)=\theta'(j+1)$. In this situation, $(\theta,\theta')=d_j(\theta\partial^{j+1},\theta'\partial^{j+1})$ is degenerate ($\partial^{j+1}d^j$ is the element $0\to\cdots\to j\to j\to j+2\to\cdots\to n$, applying θ we get $\theta(0)\to\cdots\to\theta(j)\to\theta(j)\to\cdots\to\theta(n)=\theta$ again, and similarly for θ'). Since an injective path in a grid as above has at most $p+2=|[p+1]|$ nodes, all simplices of order greater than $p+1$ are degenerate. Non-degenerate $(p+1)$-simplices are injective paths with $n+2$ nodes (maximal paths), so they are of the form $c(i)$ with $0\le i\le p$ and

$$\begin{aligned}
c(i)=&(0\to1\to\cdots\to i\to i\to i+1\to\cdots\to p, 0\to\cdots\to0\to1\to\cdots\to1)\\
=&(d^i, d^0d^1\ldots d^{i-1}d^{i+1}\ldots d^p)\in\Delta^p_{p+1}\times\Delta^1_{p+1}
\end{aligned}$$

The simplex $c(i)$ can be represented as:



Furthermore, any non-degenerate simplex h of lower order can be represented as iterated faces of a simplex $c(i)$ for some $0\le i\le p$. Indeed, there are three reasons for an injective path not to be maximal: either it does not start at $(0,0)$, or it does not end at $(p,1)$, or at one node the path moves up and right simultaneously. We complete h (in red below) into a maximal path h' (in blue below) as in the following example:



Looking at h' as a map from $[p+1]$ to $[0,p]\times[0,1]$, let $i_1<\cdots<i_{p+1-n}$ be the integers in $[p+1]$ such that $h'(i_k)$ is not a node for h. Then $h=\partial_{i_1}\dots\partial_{i_{p+1-n}}h'$. For the example above, $n=2$, $p=5$, h is represented by $(1\to 3\to 4, 0\to 1\to 1)$ and h' is $c(3)$. We have $i_1=0, i_2=2, i_3=3, i_4=6$ and $\partial_0\partial_2\partial_3\partial_6(c(3))=(d^3\partial^6\partial^3\partial^2\partial^0, d^0d^1d^2d^4d^5\partial^6\partial^3\partial^2\partial^0)=h$.

We are now ready to compare the two spaces we are interested in. Using definition (6.8) for the geometric realization,

$$|\Delta^p\times\Delta^1|=\textstyle\coprod_{n\in\mathbb{N}}(\Delta^p_n\times\Delta^1_n\times|\Delta^n|)\big/\sim_1$$

and

$$|\Delta^p|\times|\Delta^1|=\left(\textstyle\coprod_{n\in\mathbb{N}}\Delta^p_n\times|\Delta^n|\big/\sim_2\right)\times\left(\textstyle\coprod_{n\in\mathbb{N}}\Delta^1_n\times|\Delta^n|\big/\sim_3\right).$$

Let $x\in|\Delta^p\times\Delta^1|$. Using the equivalence relation $\sim_1$ and last paragraph, we may choose a non-degenerate representative for x: write $x=[c(i),(u_0,\dots,u_{p+1})]$ for some $1\le i\le p+1$. Then:

$$\begin{aligned}f(x)&=([d^i,(u_0,\dots,u_{p+1})],[d^0d^1\dots d^{i-1}d^{i+1}\dots d^p,(u_0,\dots,u_{p+1})])\\&=([d_i(\mathrm{id}_{[p]}),(u_0,\dots,u_{p+1})],[d_p\dots d_{i+1}d_{i-1}\dots d_1d_0(\mathrm{id}_{[1]}),(u_0,\dots,u_{p+1})])\\&=([\mathrm{id}_{[p]},(u_0,\dots,u_{i-1},u_i+u_{i+1},u_{i+2}\dots,u_{p+1})],[\mathrm{id}_{[1]},(u_0+\cdots+u_i,u_{i+1}+\cdots+u_{p+1})])\end{aligned}$$

corresponding to $((u_0,\dots,u_{i-1},u_i+u_{i+1},u_{i+2},\dots,u_{p+1}),(u_0+\cdots+u_i,u_{i+1}+\cdots+u_{p+1}))\in|\Delta^p|\times|\Delta^1|$.

Now we define an inverse g for f. Let $y=((u_0,\dots,u_p),(a_0,a_1))\in|\Delta^p|\times|\Delta^1|$. Let j be the smallest integer between 0 and p such that $u_0+\cdots+u_j\ge a_0$ (we have $0\le a_0\le 1$ and the left-hand side increases with j, reaching 1 at $j=p$). Let:

$$g(y):=[c(j),(u_0,\dots,u_{j-1},a_0-(u_0+\cdots+u_{j-1}),u_0+\cdots+u_j-a_0,u_{j+1},\dots,u_p)].$$

This is an element of $(\Delta^p\times\Delta^1)_{p+1}\times|\Delta^{p+1}|$, because the vector on the right-hand side has $p+2$ coordinates, all in $[0,1]$ by the choice of j, and summing to 1: indeed, since $(u_0,\dots,u_p)\in|\Delta^p|$:

$$u_0+\cdots+u_{j-1}+a_0-(u_0+\cdots+u_{j-1})+u_0+\cdots+u_j-a_0+u_{j+1}+\cdots+u_p=u_0+\cdots+u_p=1.$$

The function g is an inverse for f:

- Let $y=((u_0,\dots,u_p),(a_0,a_1))\in|\Delta^p|\times|\Delta^1|$. Then:

  $$\begin{aligned}f(g(y))&=f([(c(j),(u_0,\dots,u_{j-1},a_0-(u_0+\cdots+u_{j-1}),u_0+\cdots+u_j-a_0,u_{j+1},\dots,u_p))])\\&=([\mathrm{id}_{[p]},(u_0,\dots,u_{j-1},a_0-(u_0+\cdots+u_{j-1})+u_0+\cdots+u_j-a_0,u_{j+1},\dots,u_p)],\\&\quad[\mathrm{id}_{[1]},(u_0+\cdots+u_{j-1}+a_0-(u_0+\cdots+u_{j-1}),u_0+\cdots+u_j-a_0+u_{j+1}+\cdots+u_p)])\\&=([\mathrm{id}_{[p]},(u_0,\dots,u_{j-1},u_j,u_{j+1},\dots,u_p)],[\mathrm{id}_{[1]},(a_0,1-a_0)])\\&=([\mathrm{id}_{[p]},(u_0,\dots,u_p)],[\mathrm{id}_{[1]},(a_0,a_1)])\end{aligned}$$

  which corresponds to $y\in|\Delta^p|\times|\Delta^1|$.

- Conversely, let $x=[c(i),(u_0,\dots,u_{p+1})]\in|\Delta^p\times\Delta^1|$ (we saw above that x can be written in this form). Then

  $$f(x)=((u_0,\dots,u_{i-1},u_i+u_{i+1},u_{i+2},\dots,u_{p+1}),(u_0+\cdots+u_i,u_{i+1}+\cdots+u_{p+1})).$$

  Suppose that $u_i\ne 0$. We have $u_0+\cdots+u_{i-1}<u_0+\cdots+u_i$ but $u_0+\cdots+u_{i+1}\ge u_0+\cdots+u_i$, hence the integer j in the definition of g is i. Therefore:

  $$g(f(x))=[c(i),(u_0,\dots,u_{i-1},u_i,u_{i+1},\dots,u_{p+1})]=x.$$

Now if $u_i = 0$, suppose that $u_{i-1} \neq 0$. Then $j = i-1$ (because $u_0 + \cdots + u_{i-1} = u_0 + \cdots + u_i$ but $u_0 + \cdots + u_{i-2} < u_0 + \cdots + u_i$) and

$$\begin{aligned} g(f(x)) &= [c(i-1), (u_0, \ldots, u_{i-2}, u_{i-1} + u_i, 0, u_i + u_{i+1}, u_{i+2}, \ldots, u_{p+1})] \quad \text{(using } u_i = 0\text{)} \\ &= [c(i-1), (u_0, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{p+1})] \\ &= [\partial_i(c(i-1)), (u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{p+1})] \text{ by definition of } \sim_1 \\ &= [\partial_i(c(i)), (u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{p+1})] \qquad (\star) \\ &= [c(i), (u_0, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{p+1})] \text{ by definition of } \sim_1 \\ &= x \end{aligned}$$

The equality $(\star)$ holds because one the one hand, we have (using the cosimplicial identities):

$$\begin{aligned} \partial_i(c(i-1)) &= (d^{i-1}\partial^i, d^0 \ldots d^{i-2} d^i \ldots d^p \partial^i) \\ &= (\mathrm{id}_{[p]}, d^0 \ldots d^{i-2} d^i \ldots d^{p-1} \partial^i d^{p-1}) \\ &= (\mathrm{id}_{[p]}, d^0 \ldots d^{i-2} d^i \partial^i d^i \ldots d^{p-1}) \\ &= (\mathrm{id}_{[p]}, d^0 \ldots d^{i-2} d^i \ldots d^{p-1}) \end{aligned}$$

and on the other hand:

$$\begin{aligned} \partial_i(c(i)) &= (d^i\partial^i, d^0 \ldots d^{i-1} d^{i+1} \ldots d^p \partial^i) \\ &= (\mathrm{id}_{[p]}, d^0 \ldots d^{i-1} d^{i+1} \ldots d^{p-1} \partial^i d^{p-1}) \\ &= (\mathrm{id}_{[p]}, d^0 \ldots d^{i-2} d^{i-1} \partial^i d^i \ldots d^{p-1}) \\ &= (\mathrm{id}_{[p]}, d^0 \ldots d^{i-2} d^i \ldots d^{p-1}) \end{aligned}$$

which coincide. The situation for $u_{i-1} = u_{i-2} = 0$ is similar, iterating the same process with several faces maps.

- Then f is a continuous, bijective map between a compact space and a Hausdorff space: $|\Delta^p \times \Delta^1|$ is compact because the quotient in its definition can be rewritten as a quotient of a finite disjoint union over $n \leq p+1$ since all simplices of greater order are degenerate, all sets of simplices in $\Delta^p \times \Delta^1$ are finite, and $|\Delta^n|$ is compact for all $n \in \mathbb{N}$. The space $|\Delta^p| \times |\Delta^1|$ is Hausdorff as a product of subspaces of $\mathbb{R}^{p+1}$ and $\mathbb{R}^2$ respectively.
- Finally, we have showed that $|\Delta^p \times \Delta^1|$ is homeomorphic to $|\Delta^p| \times |\Delta^1|$. The inclusions $|\Delta^p| \to |\Delta^p \times \Delta^1|$ correspond to the usual inclusions of $|\Delta^p|$ into its cylinder: when passing to the geometric realization, the first inclusion sends any $(u_0, \ldots, u_p) \in |\Delta^p|$, corresponding to $[\mathrm{id}_{[p]}, (u_0, \ldots, u_p)]$ in the quotient, to $[(\mathrm{id}_{[p]}, \underline{0}), (u_0, \ldots, u_p)] \in |\Delta^p \times \Delta^1|$. Since

$$\partial_{p+1}(c(p)) = (d^p \partial^{p+1}, d^0 \ldots d^{p-1} \partial^{p+1}) = (\mathrm{id}_{[p]}, \partial^1 d^0 \ldots d^{p-1}) = (\mathrm{id}_{[p]}, \underline{0})$$

  we have $[(\mathrm{id}_{[p]}, \underline{0}), (u_0, \ldots, u_p)] = [c(p), (u_0, \ldots, u_p, 0)]$ by definition of $\sim_1$ and it is sent to $((u_0, \ldots, u_p), (1, 0))$ by f, as it would have been by the inclusion $|\Delta^p| \to |\Delta^p| \times |\Delta^1| \approx |\Delta^p| \times I$ into the basis of the cylinder. For the second inclusion the proof is the same, with $\underline{1}$, $c(0)$, and ∂_0 instead of $\underline{0}$, $c(p)$, and ∂_{p+1} respectively.

As explained above, this concludes the proof. □

Before stating the next lemma, we also need a definition:

Definition 6.19 (Countable simplicial maps)**.** We call a simplicial map $f : X \to Y$ *countable* if Y is countable, i.e. has only a countable number of non-degenerate simplices.

Lemma 6.20 (Fibration criterion)**.** *Let p be a simplicial map having the RLP with respect to acyclic countable cofibrations. Then p is a fibration.*

Proof. We give a sketch of a proof, relying at some point on a lemma we admit without proving it. Suppose that $p : X \to Y$ is a simplicial map having the RLP with respect to all acyclic countable cofibrations. Consider a lifting problem:

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ {\scriptstyle \iota}\big\downarrow{\scriptstyle \wr} & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[k]{} & Y \end{array}$$

The inclusion $\subseteq$ of simplicial sets defines a poset

$$\{B' \subseteq B \mid \iota(A) \subseteq B' \text{ and } p \text{ has the RLP with respect to } A \xhookrightarrow{\iota} B'\}$$

(the co-restriction of ι to B' might not be a weak equivalence, but each map induced on the sets of simplices remains injective : the co-restriction is a cofibration).

Each totally ordered subset of the poset has an upper bound: let $\{B'_i\}_{i\in I'}$ be a chain. The sequential colimit $B' := A \cup \bigcup_{i\in I'} B'_i$ exists in $\mathbf{sSet}$ by MC1. There is an inclusion $B' \subseteq B$ and the map $\iota|^{B'} : A \hookrightarrow B'$ still has the LLP with respect to p by proposition (2.10) (first natural map in a colimit). By Zorn's lemma, there exists a maximal element $B'' \subseteq B$ in the poset.

Now we show that $B'' = B$. It suffices to show that B'' contains all the non-degenerate simplices of B. Suppose for a contradiction that there exists a non-degenerate simplex $x \in B_n \setminus B''_n$ for some $n \in \mathbb{N}$. There is a diagram:

$$\begin{tikzcd}
 & A \arrow[r, "h"] \arrow[d, hook, "\iota", "\wr"'] \arrow[dr] & X \arrow[d, "p"] \\
 & & B'' \arrow[dl, hook] \\
\langle x \rangle \arrow[r, hook] & B \arrow[r, "k"'] & Y
\end{tikzcd}$$

where $\langle x \rangle$ denotes the smallest simplicial set included in B, containing x as a simplex: it contains x and all its faces and degeneracies, in particular it is countable. We admit the following lemma without proof (a proof can be found in the lecture notes by Jardine (2018) (see the bibliography (9) for more details, this is lemma 11.3), and uses the fact that the homotopy groups of the geometric realization of a countable simplicial set are countable):

Lemma 6.21 (Bounded cofibration lemma)**.** *If X, Y, and A are simplicial sets with maps:*

$$\begin{tikzcd}
 & X \arrow[d, hook, "\wr"] \\
A \arrow[r, hook] & Y
\end{tikzcd}$$

with A countable, then there exists $B \subseteq Y$ countable with $A \subseteq B$ such that the inclusion $B \cap X \to B$ is an acyclic cofibration.

Applying the lemma in our situation, there exists a countable $C \subseteq B$ with $\langle x \rangle \subseteq C$ and the inclusion $C \cap A \to C$ is an acyclic cofibration:

$$\begin{tikzcd}
 & C \cap A \arrow[r, hook] \arrow[d, hook, "\iota'"', "\wr"] & A \arrow[rr, "h"] \arrow[d, hook, "\iota"] \arrow[dr] & & X \arrow[d, "p"] \\
 & & & B'' \arrow[ur, dotted, "\ell'"] \arrow[dl, hook] & \\
\langle x \rangle \arrow[r] & C \arrow[r, hook] \arrow[uurrr, dotted, "\ell"] & B \arrow[rr, "k"'] & & Y
\end{tikzcd}$$

and ι' is countable. By assumption a lift $\ell : C \to X$ exists in the diagram above. There is also a lift $\ell' : B'' \to X$ since by hypothesis, B'' is an element of the poset previously considered. Together, ℓ and ℓ' induce a map $\ell'' : B'' \cup C \to X$. This construction shows that p has the RLP with respect to the inclusion of A into $B'' \cup C$. Since $B'' \subsetneq B'' \cup C$, this contradicts the maximality of B''. Hence we had $B'' = B$ and we are done. $\square$

6.5 Verification of the axioms

6.5.1 Axiom MC3: retracts

Suppose that $f \in \mathbf{sSet}(X, Y)$ is a retract of $g \in \mathbf{sSet}(A, B)$. There is a diagram in $\mathbf{sSet}$:

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[d, "f"'] \arrow[rr, bend left, equal] & A \arrow[r, "r_X"] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r, "\iota_Y"'] \arrow[rr, bend right, equal] & B \arrow[r, "r_Y"'] & Y
\end{tikzcd}$$

By functoriality of $|\cdot|$, when passing to the geometric realization, there is a similar diagram in **Top**, expressing $|f|$ as a retract of $|g|$. Hence if g is a weak equivalence, by definition $|g|$ is a weak equivalence in **Top**, so $|f|$ is a weak equivalence by MC3 in **Top** and f is a weak equivalence.

Since being a fibration is defined by a RLP with respect to a fixed class of maps, the class of fibrations is stable by retracts (see proposition (2.10)).

Suppose now that g is a cofibration instead. Since the composition of simplicial maps is defined pointwise, we have the same diagram on each "coordinate": for all $n \in \mathbb{N}$, f_n is a retract of g_n and $(r_X)_n \circ (\iota_X)_n = (\mathrm{id}_X)_n = \mathrm{id}_{X_n}$ implying that $(\iota_X)_n$ is injective (similarly $(\iota_Y)_n$ is injective). Then $g_n \circ (\iota_X)_n = (\iota_Y)_n \circ f_n$ is injective implying that f_n is injective. Hence f is a cofibration.

Therefore axiom MC3 holds.

6.5.2 Axiom MC5: factorization

As in the category **Top**, we begin by constructing factorizations of maps and then use them to verify axiom MC4.

Factorization as a cofibration and an acyclic fibration. Let $f : X \to Y$ be a simplicial map. In view of the criterion provided by lemma (6.17), we proceed by a small object argument on the set of maps $B = \{b_n : \partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$. Reproducing the proof we did in **Top**, we consider the pushout (exists by MC1):

$$\begin{tikzcd}
\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} \partial\Delta^n \arrow[r, "\sum_{n\in\mathbb{N}}\sum_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} g"] \arrow[d, "\coprod_{n\in\mathbb{N}}\coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} b_n"'] & X \arrow[d, "c_1"] \\
\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} \Delta^n \arrow[r, "e_1"'] & G^1(B,f) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By the universal property of the pushout, the maps $\sum_{n\in\mathbb{N}}\sum_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} h$ and $f : X \to Y$ induce a map $g_1 : G^1(B,f) \to Y$. Inductively we define $G^\ell(B,f) = G^1(B, g_{\ell-1})$ for all $\ell \geq 2$ and get maps $c_\ell : G^{\ell-1}(B,f) \to G^\ell(B,f)$, $e_\ell : \coprod_{n\in\mathbb{N}}\coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,c_{\ell-1})} \Delta^n \to G^\ell(B,f)$ and $g_\ell : G^\ell(B,f) \to Y$ (with $G^0(B,f) = X$).

Let $G^\infty(B,f)$ be the sequential colimit (by MC1) of the diagram

$$X \xrightarrow{c_1} G^1(B,f) \xrightarrow{c_2} G^2(B,f) \xrightarrow{c_3} \dots$$

There is a natural map $\iota : X \to G^\infty(B,f)$, and by the universal property of the colimit, the maps $\{g_\ell\}_{\ell\in\mathbb{N}}$ induce a map $p : G^\infty(B,f) \to Y$ with $p \circ \iota = f$.

Now we check that ι is a cofibration. First of all, c_ℓ is a cofibration for all $\ell \in \mathbb{N}^*$. Since colimits are computed pointwise in $_{\mathbf{S}}\mathbf{Set}$, for all $m \in \mathbb{N}$ there is a pushout diagram in **Set**:

$$\begin{tikzcd}
\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} (\partial\Delta^n)_m \arrow[r, "u_m:=\sum_{n\in\mathbb{N}}\sum_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} g_m"] \arrow[d, "v_m:=\coprod_{n\in\mathbb{N}}\coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} (b_n)_m"'] & X_m \arrow[d, "(c_1)_m"] \\
\coprod_{n\in\mathbb{N}} \coprod_{(g,h)\in_{\mathbf{S}}\mathbf{Set}^{\to}(b_n,f)} (\Delta^n)_m \arrow[r, "(e_1)_m"'] & G^1(B,f)_m \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By definition of pushouts in **Set**, if $(c_1)_m(x) = (c_1)_m(x')$ for some $x, x' \in X_m$, there exist two points y, y' with $u_m(y) = x$ and $u_m(y') = x'$, such that $v_m(y) = v_m(y')$ so x, x' are identified with $v_m(y) = v_m(y')$ in the colimit. But v_m is injective as a disjoint union of the injective maps $(b_n)_m$ (b_n is a cofibration for all $n \in \mathbb{N}$), so $y = y'$ and $x = x'$. Hence $(c_1)_m$ is injective. The situation is similar for the other maps c_k, $k \in \mathbb{N}^*$: they are all cofibrations, and can be seen as inclusions. Then ι itself is a cofibration: again because limits are computed pointwise, for all $n \in \mathbb{N}$, we have $(G^\infty(B,f))_n = X_n \cup \bigcup_{k\in\mathbb{N}^*} G^k(B,f)_n$ so that $\iota_n : X_n \to (G^\infty(B,f))_n$ is just the inclusion.

Finally, we check that p has the desired RLP. Consider a map $b_n \in B$ and a lifting problem:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "g"] \arrow[d, hook, "b_n"'] & G^\infty(B,f) \arrow[d, "p"] \\ \Delta^n \arrow[r, "h"'] & Y \end{tikzcd}$$

The map g factors through $G^k(B,f)$ for some $k \geq 1$: indeed $\partial\Delta^n$ has a finite number of non-degenerate simplices (by definition, all m-simplices for $m \geq n$ are degenerate, and the m-simplices for $m < n$ are order preserving maps between $[m]$ and $[n]$, there are only finitely many of them). Let $\{a_1, \ldots, a_m\}$ be the finite set of non degenerate simplices of $\partial\Delta^n$, with a_i a k_i-simplex for all $i \leq m$. For all $i \leq m$, there exists an integer k'_i such that $g_{k_i}(a_i) \in G^{k'_i}(B,f)_{k_i}$. Set $k = \max\{k'_1, \ldots, k'_m\}$. Then $g(\partial\Delta^n) \subseteq G^k(B,f)$ (all simplices in $\partial\Delta^n$ other than $\{a_1, \ldots, a_m\}$ are obtained via degeneracies, hence their image by g is obtained via a degeneracy in $G^k(B,f)$ of simplices in $\{g(a_1), \ldots, g(a_m)\}$).

Therefore, by the same argument as in **Top**, we have a diagram:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "g"] \arrow[d, "b_n"'] & G^k(B,f) \arrow[r, "c_{k+1}"] \arrow[d, "g_k"'] & G^{k+1}(B,f) \arrow[r, "\tilde{c}_{k+1}"] \arrow[d, "g_{k+1}"'] & G^\infty(B,f) \arrow[d, "p"] \\ \Delta^n \arrow[r, "h"'] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

with $(g,h) \in \mathbf{sSet}^{\to}(b_n, g_k)$, so h lifts into a map $\ell : \Delta^n \to G^{k+1}(B,f)$ by construction, and the composition $\tilde{c}_{k+1} \circ \ell$ is a solution to our original lifting problem.

We have obtained a factorization $X \xrightarrow{\iota} G^\infty(B,f) \xrightarrow{p} Y$ of f, where ι is a cofibration and p has RLP with respect to all maps in B. In particular, by lemma (6.17), p is an acyclic fibration. We can set $\omega'(f) = G^\infty(B,f)$, $\iota'(f) = \iota$ and $\pi'(f) = p$.

Factorization as an acyclic cofibration and a fibration. This time we proceed by a transfinite small object argument. In view of the criterion provided by lemma (6.20), we consider the set of all countable acyclic cofibrations. Choose a cardinal number κ, interpreted as the poset of ordinal numbers $\lambda < \kappa$ (κ is the set of its predecessors). We do the construction for any choice of κ, and at the end we will see that choosing $\kappa = \aleph_1$ suffices to conclude the proof. We will build factorizations:

$$\begin{tikzcd} X \arrow[r, "\iota_\lambda"] \arrow[dr, "f"'] & Z_\lambda \arrow[d, "p_\lambda"] \\ & Y \end{tikzcd}$$

for all $\lambda < \kappa$, where ι_λ is built from acyclic cofibrations by the operations described in proposition (2.10). Consider $Z_0 = X$, $\iota_0 = \mathrm{id}_X$, and $p_0 = f$.

- There is a set C of countable acyclic cofibrations, seen as inclusions, with simplices in $\mathbb{N}$: for all such cofibration c, the data of the target B_c amounts to choosing for all $n \in \mathbb{N}$ a subset $B_n \subseteq \mathbb{N}$, and choosing a finite number of faces and degeneracies between B_n and B_{n+1} for all $n \in \mathbb{N}$. The choice for the source A_c is the same except that we have additional conditions due to the fact that A must be included in B. Then c is uniquely determined by the data of the map it induces on n-simplices. So C is indeed a set. And for all $c \in C$ and fixed simplicial map q, $\mathbf{sSet}^{\to}(c, q)$ is a set too. We need this because a priori there are only small colimits in $\mathbf{sSet}$.
- <u>Successor:</u> Suppose that a factorization as above is already defined for some $\lambda < \kappa$. We want

to define a factorization for $s = \lambda + 1$. Let $Z_{\lambda+1}$ be the pushout:

$$\begin{tikzcd}
\coprod_{c\in C}\coprod_{(g,h)\in \mathbf{sSet}^{\to}(c,p_\lambda)} A_c \arrow[rr, "\sum_{c\in C}\sum_{(g,h)\in \mathbf{sSet}^{\to}(c,p_\lambda)} g"] \arrow[d, "\coprod_{c\in C}\coprod_{(g,h)\in \mathbf{sSet}^{\to}(c,p_\lambda)} c"'] & & Z_\lambda \arrow[d, "\iota'_{\lambda+1}"] \arrow[ddr, bend left, "p_\lambda"] & \\
\coprod_{c\in C}\coprod_{(g,h)\in \mathbf{sSet}^{\to}(c,p_\lambda)} B_c \arrow[rr] \arrow[drrr, bend right, "\sum_{c\in C}\sum_{(g,h)\in \mathbf{sSet}^{\to}(c,p_\lambda)} h"'] & & Z_{\lambda+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! p_{\lambda+1}"] & \\
 & & & Y
\end{tikzcd}$$

and $p_{\lambda+1}$ is obtained by the universal property. In this situation $\iota'_{\lambda+1}$ is an acyclic cofibration: by definition the acyclic cofibrations in $\mathbf{sSet}$ are exactly the maps having the LLP with respect to fibrations. Since $\iota'_{\lambda+1}$ was built from acyclic cofibrations, having this LLP, by the operations of proposition (2.10), $\iota'_{\lambda+1}$ has this LLP and is an acyclic cofibration. Then $\iota_{\lambda+1} := \iota'_{\lambda+1} \circ \iota_\lambda$ is also an acyclic cofibration by composition and hypothesis of induction. By construction, we have $p_{\lambda+1} \circ \iota_{\lambda+1} = p_{\lambda+1} \circ \iota'_{\lambda+1} \circ \iota_\lambda = p_\lambda \circ \iota_\lambda = f$, as required.

- Limit: If $\gamma < \kappa$ is a limit ordinal, assume that we already defined a factorization for all ordinals $\lambda < \gamma$. Let $Z_\gamma = \mathrm{colim}_{\lambda<\gamma} Z_\lambda$ (this is a colimit over the category associated to the poset of the ordinals preceding γ (γ is the set of its predecessors)). There is a natural map $\iota_\gamma : Z_0 = X \to Z_\gamma$ and the universal property of the colimit applied to the maps $\{p_\lambda\}_{\lambda<\gamma}$ gives a map $p_\gamma : Z_\gamma \to Y$, satisfying $p_\gamma \circ \iota_\gamma = p_0 = f$ as required. The map ι_γ is an acyclic cofibration for the same reason as above.

- Finally, we repeat the limit step for $\gamma = \kappa$. We obtain a simplicial set $Z_\kappa =: Z$ and maps $\iota : X \to Z$, $p : Z \to Y$ with $p \circ \iota = f$.

The map ι is an acyclic cofibration for the same reason as above. To conclude the proof by lemma (6.20), we have to show that p has the RLP with respect to any countable acyclic cofibration. Let $j : A \to B$ be such a cofibration. Then, j may be seen as an inclusion, and since by assumption B is countable, we may, up to relabelling its simplices (choose a bijection between B_n and $\mathbb{N}$ for all $n \in \mathbb{N}$), consider that B has simplices in $\mathbb{N}$ and that j is an element of C. In this case, consider a lifting problem:

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, hook, "j"', "\wr"] & Z \arrow[d, "p"] \\
B \arrow[r, "\ell"'] & Y
\end{tikzcd}$$

We would like to solve the lifting problem in Z_λ for some $\lambda < \kappa$. For this, we need to ensure that all the images by k of non-degenerate simplices of A are contained in Z_λ. Since A is countable, choosing $\kappa = \aleph_1$ at the beginning of the proof is enough: there is an uncountable number of simplicial sets in the colimit $\mathrm{colim}_{\gamma<\kappa} Z_\gamma = Z_\kappa$, so such an ordinal λ exists. Therefore, by construction, we can solve the lifting problem exactly as for the first factorization with Z_λ and $Z_{\lambda+1}$ (κ is a limit ordinal so $\lambda + 1 < \kappa$). Finally, we can set $\omega(f) = Z$, $\iota(f) = \iota$ and $\pi(f) = p$.

6.5.3 Axiom MC4: lifting problems

The idea is the same as in **Top**. Fibrations have the RLP with respect to acyclic cofibrations by definition. Cofibrations have the LLP with respect to acyclic fibrations: indeed, let $p : X \overset{\sim}{\twoheadrightarrow} Y$ be an acyclic fibration. We will show that p has the RLP with respect to all inclusions $\partial\Delta^n \to \Delta^n$ for $n \in \mathbb{N}$: we have seen in the proof of lemma (6.17) that if p satisfies this condition, then it has the RLP with respect to all cofibrations.

Since the factorization axiom holds, we can write p as a composition $X \overset{\iota'}{\hookrightarrow} Z \underset{\sim}{\overset{p'}{\twoheadrightarrow}} Y$, where p' was built (see the proof of MC5) as having the RLP with respect to all inclusions $\partial\Delta^n \to \Delta^n$ for $n \in \mathbb{N}$ and is an acyclic fibration. By the 2 of 3 rule we already proved, ι' is a weak equivalence too. By definition of a fibration, a lift exists in the first diagram below, allowing us to express p as a retract of p' (second diagram):

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, hook, "\iota'"', "\wr"] & X \arrow[d, two heads, "p", "\wr"'] \\
Z \arrow[r, two heads, "p'", "\sim"'] \arrow[ur, dashed, "\ell"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\iota'"'] \arrow[rr, bend left, equal] \arrow[d, "p"] & Z \arrow[r, "\ell"'] \arrow[d, "p'"] & X \arrow[d, "p"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Applying proposition (2.10) once more to the set of maps $S = \{b_n : \partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$, we get that p has the RLP with respect to the maps in S too, so we are done.

This finishes the proof of theorem (6.15).

7 Equivalence of the homotopy categories of Top and sSet

In this section we apply theorem (4.4) to the adjunction $|\cdot| \dashv S$ between $\mathbf{_SSet}$ and $\mathbf{Top}$, with the Quillen model structures described in sections (6) and (5) respectively, in order to obtain an adjunction and an equivalence of categories between their homotopy categories $\mathrm{Ho}(\mathbf{Top})$ and $\mathrm{Ho}(\mathbf{_SSet})$.

7.1 The homotopy groups for simplicial sets

For this subsection, we follow the book "Model categories" by Hovey (1999). To check that the conditions of theorem (4.4) hold, we will have to show that some maps are weak equivalences, and for this we need to compare the homotopy groups of some topological spaces. A useful intermediary in this comparison is given by the corresponding notion for simplicial sets: the simplicial homotopy groups. Indeed, if we make sense of the n-th homotopy group $\pi_n(Z, v)$ for Z a (fibrant) simplicial set and v a vertex of Z (i.e. a 0-simplex $v \in Z_0$), we have the following result, which we will not prove:

Theorem 7.1 (Simplicial and topological homotopy groups)**.** *Let Z be a fibrant simplicial set, and $v \in Z_0$ a 0-simplex in Z. Then, there is an isomorphism (a bijection of pointed sets when $n = 0$):*

$$\pi_n(Z, v) \cong \pi_n(|Z|, |v|)$$

where $|v|$ denotes the point representing the vertex v in the geometric realization $|Z|$.

Proof. For a proof, see the book by Hovey (1999) (this is Proposition 3.6.3, p 97). The proof goes by induction on n and uses the long exact homotopy sequences both for simplicial sets and topological spaces. We will show the case $n = 0$ as an example in this subsection (see lemma (7.3)). $\square$

Let Z be a fibrant simplicial set and v be a vertex of Z. The group structure on $\pi_n(Z, v)$ comes from the bijection in theorem (7.1). So we only define *sets* $\pi_n(Z, v)$ for all $n \in \mathbb{N}$.

To begin with, we are interested in defining the set $\pi_0(Z, v)$ of path components:

Definition 7.2 (Homotopy of 0-simplices and path components)**.** Let $x, y \in Z_0$ be 0-simplices of the fibrant simplicial set Z. We write $x \sim y$ and say that x and y are *homotopic* if there exists a 1-simplex $z \in Z_1$ with faces x and y, i.e. $\partial_0(z) = x$ and $\partial_1(z) = y$. Then, the relation $\sim$ is an equivalence relation on Z_0. The *set of (simplicial) path components of* Z (with basepoint $v \in Z_0$) is defined as the pointed set (with basepoint $[v]$) of homotopy classes of 0-simplices of Z, and it is denoted by $\pi_0(Z, v)$.

We will prove that homotopy of vertices is an equivalence relation in the more general case of n-simplices.

Lemma 7.3 (Case $n = 0$ in theorem (7.1))**.** *There is a bijection of pointed sets:*

$$\pi_0(Z, v) \cong \pi_0(|Z|, |v|).$$

where $|v|$ denotes the point representing the 0-simplex v in the geometric realization $|Z|$. On the left hand side, π_0 is the set of simplicial path components, whereas on the right hand side it is taken in the topological sense.

Proof. Let $[f] \in \pi_0(Z, v)$ be a path component with representative $f \in Z_0$. Then f corresponds to a point $|f|$ in the geometric realization (definition (6.8)):

$$|Z| = \left(\coprod_{n\in\mathbb{N}}(Z_n \times |\Delta^n|)\right)\Big/\approx$$

because we can choose the equivalence class of the point $|\Delta^0|$ indexed by f in the disjoint union (this is the point that we call $|f|$). Now take the (topological) path component of this point. This gives a well-defined map $\varphi : \pi_0(Z, v) \to \pi_0(|Z|, |v|)$: suppose that x and y are homotopic 0-simplices of Z. Then, there is a 1-simplex $z \in Z_1$ with $\partial_0(z) = x$ and $\partial_1(z) = y$ by hypothesis. Therefore, mapping the interval I to the topological 1-simplex indexed by z in the geometric realization gives a path from $|x|$ to $|y|$ in $|Z|$: indeed the definition of the relation $\approx$ and the conditions $\partial_0(z) = x$, $\partial_1(z) = y$ imply that $|x|$ and $|y|$ are identified with the endpoints of the 1-simplex indexed by z in the quotient.

The map φ is surjective: consider any path component $[x] \in \pi_0(|Z|, |v|)$ with representative a point $x \in |Z|$, which in turns admits a representative x' in some topological standard simplex $|\Delta^n|$ indexed an n-simplex $x_n \in Z_n$. Since $|\Delta^n|$ is path-connected, x' is in the same path component as some point x'' in $|\Delta^n|$ with all coordinates equal to 0 except one. The latter corresponds via the relation $\approx$ to a topological 0-simplex $|\Delta^0|$ indexed by a 0-simplex in Z. Because we stayed in the same path component while doing these choices, the path component of this 0-simplex has image $[x]$.

The map φ is injective: for $[x] \in \pi_0(Z, v)$ consider the set $Z_{[x]}$ of all simplices in Z having a vertex in $[x]$ (i.e. simplices for which, by taking successive faces, we can find a 0-simplex homotopic to x). Then $Z_{[x]}$ is itself a simplicial set, included in Z (indeed, $Z_{[x]}$ is stable under faces by construction and under degeneracies because of the simplicial identity $\partial_j d_j = \mathrm{id}$: up to taking one face more we can cancel the degeneracy operation). With this notation Z can be written as $\coprod_{[x]\in\pi_0(Z,v)} Z_{[x]}$. To justify this, we check on each set of n-simplices that the union is disjoint. Let $n \in \mathbb{N}$ and assume that $z \in (Z_{[x]})_n \cap (Z_{[y]})_n$. Then, z is an n-simplex (a map $z : \Delta^n \to Z$) with vertices x' and y' (maps $x' : \Delta^0 \xrightarrow{\tilde{x}} \Delta^n \xrightarrow{z} Z$ and $y' : \Delta^0 \xrightarrow{\tilde{y}} \Delta^n \xrightarrow{z} Z$) homotopic to x and y respectively. There is some face of order 1 of z with faces x' and y': indeed, if $\tilde{x}_0(\mathrm{id}_{[0]})$ is the map $[0] \to [n]$ with image $\{k\}$ and $\tilde{y}_0(\mathrm{id}_{[0]})$ is the map $[0] \to [n]$ with image $\{m\}$, consider the simplicial map $\Delta^1 \to \Delta^n$ sending $\mathrm{id}_{[1]}$ to the order preserving map $[1] \to [n]$ with image $\{k, m\}$ and post compose it by z. Hence x' is homotopic to y' and $[x] = [x'] = [y'] = [y]$. Since geometric realization preserves colimits, $|Z|$ is the disjoint union $\coprod_{[x]\in\pi_0(Z,v)} |Z_{[x]}|$. Any $[x] \in \pi_0(Z, v)$ is sent to a point in $|Z_{[x]}|$, so all distinct points in $\pi_0(Z, v)$ must be sent to distinct path components in $|Z|$. □

Now we define higher simplicial homotopy groups:

Definition 7.4 (Homotopy of n-simplices and higher homotopy groups)**.** Let Z be a fibrant simplicial set, $n \in \mathbb{N}$ and $x, y \in Z_n$ be two n-simplices of Z_n with boundary in v (i.e. simplicial maps $\Delta^n \to Z$ sending all of $\partial\Delta^n$ to iterated degeneracies of v). They are called *homotopic* if there exists a simplicial map (a *(simplicial) homotopy*) $H : \Delta^n \times \Delta^1 \to Z$ such that H is equal to x on $\Delta^n \times \partial_0(\mathrm{id}_{[1]})$, to y on $\Delta^n \times \partial_1(\mathrm{id}_{[1]})$ and constant equal to v (and its degeneracies) on $\partial\Delta^n \times \Delta^1$. This defines an equivalence relation on the set of such simplices, the set of its equivalence classes is denoted by $\pi_n(Z, v)$, and is called the *n-th (simplicial) homotopy group* of Z (with basepoint v).

The group structure is given by theorem (7.1). It makes sense to talk about x and y in this definition as maps $\Delta^n \to Z$, since we have seen that by the Yoneda lemma, n-simplices (elements in Z_n) are in (natural) bijection with the simplicial maps $\Delta^n \to Z$ (see remark (6.10)). This definition is very similar to its topological version: a sphere is nothing but a disk with its boundary collapsed in one point, so maps from the sphere can be seen as maps from the disk that are constant on the boundary, and this corresponds to asking for maps from Δ^n that are constant on $\partial\Delta^n$. Considering the pushout:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, hook, "b_n"] \arrow[d] & \Delta^n \arrow[d] \\
\Delta^0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^n
\end{tikzcd}$$

we obtain a "*simplicial n-sphere*" S^n and by the universal property of the pushout, the maps from S^n to Z are exactly the maps from Δ^n to Z that are constant on $\partial\Delta^n$. In the situation of definition

(7.4), simplicial homotopies corresponds to maps $S^n \times \Delta^1$ that are equal to x and y respectively on $S^n \times \partial_0(\mathrm{id}_{[1]})$ and $S^n \times \partial_1(\mathrm{id}_{[1]})$.

The definition (7.4) corresponds to definition (7.2) in the case $n = 0$: indeed, $\partial\Delta^0$ is the empty simplicial set, so all 0-simplices of Z can be seen as maps $\Delta^0 \to Z$ that are constant on the boundary. A homotopy becomes a map $\Delta^0 \times \Delta^1 \to Z$. Since Δ^0 has a unique non-degenerate simplex and behaves like a point, this is just a map $\Delta^1 \to Z$, i.e. a 1-simplex, and the condition that H is equal to x, respectively y, at the two endpoints $\Delta^0 \times \partial_0(\mathrm{id}_{[1]})$, respectively $\Delta^0 \times \partial_1(\mathrm{id}_{[1]})$, corresponds exactly to the condition that the 1-simplex in question has x and y as faces.

We have to check that homotopy is an equivalence relation on $\mathbf{sSet}(S^n, Z)$. To do so, we will use the definitions and results of subsection (3.2.1), in which we discussed cylinder objects and left-homotopy in general model categories. We have seen in lemma (3.7) that left-homotopy is an equivalence relation when the source object is cofibrant. Since the initial object in $\mathbf{sSet}$ is the empty simplicial set (computing the colimit pointwise in **Set**, each set of simplices must be empty), and that its inclusion into any simplicial set is injective, any simplicial set is cofibrant. However, recall that the left-homotopy equivalence relation was defined through arbitrary cylinder objects. But we saw (remark (3.22)) that if the target object, here Z, was fibrant and the source object, here S^n, was cofibrant, two maps $S^n \to Z$ were left-homotopic if and only if they were homotopic through some *fixed* good cylinder object for S^n. The product $S^n \times \Delta^1$ is a good cylinder object for S^n: indeed, there are inclusions $S^n = S^n \times \partial_1(\mathrm{id}_{[1]}) \hookrightarrow S^n \times \Delta^1$ and $S^n = S^n \times \partial_0(\mathrm{id}_{[1]}) \hookrightarrow S^n \times \Delta^1$, inducing an inclusion $S^n \amalg S^n \hookrightarrow S^n \times \Delta^1$ and the projection $S^n \times \Delta^1 \to S^n$ on the first component is a weak equivalence: in **Top**, the projection $|S^n \times \Delta^1| \approx |S^n| \times I \to |S^n|$ is a homotopy equivalence (see the proof of lemma (6.17) for the identification $|S^n \times \Delta^1| \approx |S^n| \times I$), in particular it is a weak homotopy equivalence. Moreover, the composition of this projection with the inclusions is equal to $\mathrm{id}_{S^n} + \mathrm{id}_{S^n}$ as required. Simplicial homotopies in definition (7.4) correspond to the left-homotopies defined in this context. Hence the simplicial homotopy relation is exactly the left-homotopy equivalence relation on $\mathbf{sSet}(S^n, Z)$ we defined earlier in the context of general model categories.

With this connection to subsection (3.2.1), we also get other results, for example remark (3.13) implies that any simplicial map of fibrant simplicial sets $f : Z \to Z'$ induces by post-composition a map $\pi_n(Z, v) \to \pi_n(Z', f(v))$.

7.2 Quillen adjunction

We will now verify the assumptions of theorem (4.4), implying that $|\cdot| \dashv \mathbf{Top}$ is a Quillen adjunction.

Weak equivalences. By definition of the weak equivalences in $\mathbf{sSet}$ in theorem (6.15), the geometric realization functor preserves weak equivalences.

Cofibrations. Let $f : X \hookrightarrow Y$ be a cofibration between two simplicial sets. Then f can be seen as an inclusion. We want to show that $|f| : |X| \to |Y|$ is a cofibration in **Top**, i.e. that it has the LLP with respect to acyclic Serre fibrations. By lemma (5.5) we know that acyclic Serre fibrations have the RLP with respect to relative CW-inclusions. But $|f|$ is such a map: recall definition (6.18) and the proof of lemma (6.17), where we obtained a description of any inclusion of simplicial sets $X \hookrightarrow Y$ as a sequential colimit:

$$X \hookrightarrow X \cup \mathrm{sk}_0 Y \hookrightarrow X \cup \mathrm{sk}_1 Y \hookrightarrow \dots$$

and pushout diagrams:

$$\begin{tikzcd}
\coprod_{x \in Y_i \setminus X_i} \partial\Delta^i \arrow[r, "\sum_{x \in Y_i \setminus X_i} \gamma_i(x)|_{\partial\Delta^i}"] \arrow[d, "\coprod_{x \in Y_i \setminus X_i} b_i"'] & X \cup \mathrm{sk}_{i-1} Y \arrow[d] \\
\coprod_{x \in Y_i \setminus X_i} \Delta^i \arrow[r, "\sum_{x \in Y_i \setminus X_i} \gamma_i(x)"'] \arrow[ur, phantom, "\ulcorner", very near end] & X \cup \mathrm{sk}_i Y
\end{tikzcd}$$

where $\gamma_i : Y_i \to {}_{\mathbf{S}}\mathbf{Set}(\Delta^i, Y)$ is the Yoneda bijection from remark (6.10). Since the geometric realization functor is a left adjoint and preserves colimits, we have a sequential colimit:

$$\begin{tikzcd}
{|X|} \arrow[r, "\iota_0"] \arrow[rrd, "{|f|}"'] & {|X \cup \mathrm{sk}_0 Y|} \arrow[r, "\iota_1"] \arrow[rd] & {|X \cup \mathrm{sk}_1 Y|} \arrow[r, "\iota_2"] \arrow[d] & \dots \\
 & & {|Y|} &
\end{tikzcd}$$

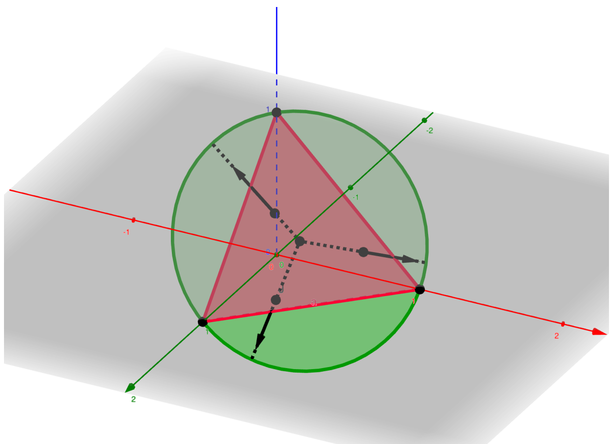
with ι_k for $k \in \mathbb{N}$ obtained as the pushout:

$$\begin{tikzcd}
\coprod_{x \in Y_i \setminus X_i} |\partial \Delta^i| \arrow[r, "\sum_{x \in Y_i \setminus X_i} |\gamma(x)|_{\partial \Delta^i}|"] \arrow[d, "\coprod_{x \in Y_i \setminus X_i} |b_i|"'] & {|X \cup \mathrm{sk}_{i-1} Y|} \arrow[d, "\iota_i"] \\
\coprod_{x \in Y_i \setminus X_i} |\Delta^i| \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & {|X \cup \mathrm{sk}_i Y|}
\end{tikzcd}$$

This implies that $|X \cup \mathrm{sk}_i Y|$ is obtained from $|X \cup \mathrm{sk}_{i-1} Y|$ by attaching i-cells: indeed

$$|\Delta^i| = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \middle| x_j \geq 0 \ \forall j \leq n \text{ and } \sum_{j=0}^{n} x_j = 1 \right\} \approx \mathbb{D}^i$$

and $|\partial \Delta^i| \approx \mathbb{S}^i$. To see this, consider the i-disk with boundary the $(i-1)$-sphere circumscribed to the standard topological i-simplex. A homeomorphism "slides" each point along the radius of the sphere that contains this point. Another way to see better the link between $|\Delta^i|$ and $\mathbb{D}^i$ is to note that there is a homeomorphism between $|\Delta^i|$ and the part of the i-disk intersecting $(\mathbb{R}^+)^i$, mapping a point $(x_0, x_1, \dots, x_i) \in \Delta^i$ to $(\sqrt{x_1}, \dots, \sqrt{x_i}) \in \mathbb{D}^i \cap (\mathbb{R}^+)^i$. The boundary is mapped to the boundary: the (topological) boundary $\partial|\Delta^i|$ is the set of points $(x_0, \dots, x_i) \in |\Delta^i|$ with $x_j = 0$ for some $j \leq i$. The homeomorphism described above maps any point with $x_j = 0$ for $j \geq 1$ to a point with a zero coordinate, on the boundary of $\mathbb{D}^i \cap (\mathbb{R}^+)^i$ and if $x_0 = 0$, then it is mapped to a point with unit norm, still on the boundary. Moreover, the map $|b_i|$ corresponds to the inclusion $\mathbb{S}^{i-1} \to \mathbb{D}^i$: the points in $|\Delta^i|$ with a zero-coordinate, forming the topological boundary, corresponds to the identification $(\partial_j(x_i), (u_0, \dots, u_{i-1})) \sim (x_i, (u_0, \dots, u_{j-1}, 0, u_j, u_{j+1}, \dots, u_{i-1}))$ in definition (6.8) of the geometric realization (the simplices of $\partial\Delta^i$ are faces of the simplices of Δ^i). In the case $i = 2$ the situation is illustrated by:



Hence $|f| : |X| \to |Y|$ is a relative CW-complex inclusion and has the LLP with respect to acyclic Serre fibrations. Finally, the functor $|\cdot|$ preserves cofibrations, and since it also preserves weak equivalences, it preserves acyclic cofibrations. Then, the adjunction $|\cdot| \dashv S$ is a Quillen adjunction, it induces an adjunction between the homotopy categories $\mathrm{Ho}(\mathbf{Top})$ and $\mathrm{Ho}({}_{\mathbf{S}}\mathbf{Set})$.

7.3 Quillen equivalence

By theorem (4.4), to show that $\mathrm{Ho}(\mathbf{Top})$ and $\mathrm{Ho}({}_{\mathbf{S}}\mathbf{Set})$ are equivalent categories, it suffices to show that for any cofibrant simplicial set X and fibrant topological space Y, the natural bijection

$\alpha_{X,Y} : \mathbf{Top}(|X|, Y) \to \mathbf{sSet}(X, S(Y))$ corresponding to the adjunction $|\cdot| \dashv S$ of proposition (6.14) preserves and creates weak equivalences. We do a sketch of a proof here. Let $f \in \mathbf{Top}(|X|, Y)$. We have to show that f is a weak equivalence in $\mathbf{Top}$ if and only if $\alpha_{X,Y}(f)$ is a weak equivalence of simplicial sets, i.e. if and only if $|\alpha_{X,Y}(f)|$ is a weak equivalence in $\mathbf{Top}$. Let $\varepsilon : |\cdot| \circ S \to \mathrm{id}_{\mathbf{Top}}$ be the co-unit and $\eta : \mathrm{id}_{\mathbf{sSet}} \to S \circ |\cdot|$ be the unit associated to this adjunction. We have $\alpha_{X,Y}(f) = S(f) \circ \eta_X$, so by naturality of ε and the triangular identities:

$$\varepsilon_Y \circ |\alpha_{X,Y}(f)| = \varepsilon_Y \circ |S(f)| \circ |\eta_X| = f \circ \varepsilon_{|X|} \circ |\eta_X| = f.$$

Therefore, by the 2 of 3 rule in $\mathbf{Top}$, it suffices to show that $\varepsilon_Y : |S(Y)| \to Y$ is a weak equivalence, namely that ε_Y induces isomorphisms $\pi_n(|S(Y)|, y) \cong \pi_n(Y, \varepsilon_Y(y))$ for all $n \in \mathbb{N}$ and $y \in |S(Y)|$.

Since conditions (i),(ii), and (iii) in theorem (4.4) are equivalent, the functor S preserves fibrations and limits, in particular it preserves fibrant objects. Let $n \in \mathbb{N}$. Then $S(Y)$ is fibrant in $\mathbf{sSet}$, and by theorem (7.1), there is an isomorphism $\pi_n(S(Y), v) \cong \pi_n(|S(Y)|, |v|)$ at any basepoint $v \in S(Y)_0$. Composing the homomorphism $\pi_n(|S(Y)|, y) \to \pi_n(Y, \varepsilon_Y(y))$ induced by ε_Y by this isomorphism, it suffices to show that the homomorphism $\pi_n(S(Y), v) \to \pi_n(Y, y)$ (for suitable basepoints $y \in Y$ and $v \in S(Y)_0$) is bijective. The simplicial n-sphere was defined as the pushout:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, hook, "b_n"] \arrow[d] & \Delta^n \arrow[d] \\ \Delta^0 \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Since the geometric realization functor is a left adjoint, it preserves pushouts, so the diagram:

$$\begin{tikzcd} |\partial\Delta^n| \arrow[r, "|b_n|"] \arrow[d] & |\Delta^n| \arrow[d] \\ |\Delta^0| \arrow[r] & |S^n| \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

is a pushout: it corresponds to attaching an n-cell to point, i.e. $|S^n| \approx \mathbb{S}^n$.

The homomorphism $\varphi : \pi_n(S(Y), v) \to \pi_n(Y, y)$ above admits an inverse: any element of $\pi_n(Y, y)$ is represented by a map $\mathbb{S}^n = |S^n| \to Y$ in $\mathbf{Top}$ sending the basepoint to y, under the adjunction it gives a simplicial map $S^n \to S(Y)$. This process preserves homotopy: indeed, there is a natural bijection

$$\mathbf{sSet}(S^n \times \Delta^1, S(Y)) \simeq \mathbf{Top}(|S^n \times \Delta^1|, Y) \simeq \mathbf{Top}(\mathbb{S}^n \times I, Y)$$

because we saw in the proof of lemma (6.17) that $|S^n \times \Delta^1| \approx |S^n| \times I \approx \mathbb{S}^n \times I$. Hence the construction passes to homotopy groups and defines an inverse for φ (all these maps are induced by the adjunction).

Finally, ε_Y is a weak equivalence. As explained above, this finishes the proof of the following:

Theorem 7.5 (Quillen equivalence between $\mathbf{Top}$ and $\mathbf{sSet}$)**.** *The adjunction*

$$|\cdot| : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : S$$

between the categories of simplicial sets and topological spaces, both with the Quillen model structure, is a Quillen equivalence.

Therefore, we obtained an adjunction and an equivalence of categories between $\mathrm{Ho}(\mathbf{Top})$ and $\mathrm{Ho}(\mathbf{sSet})$. Simplicial sets constitute good "models" for topological spaces. As seen in subsection (4.2), the (total) left and right derived functors composing the equivalence involve fibrant, respectively cofibrant replacements, that are weakly equivalent to the original objects. Up to these replacements, there is a correspondence between spaces and simplicial sets, so that some topological problems in homotopy may be treated thanks to simplicial sets (and vice-versa), by a translation from the "continuous" world of topological spaces to the "combinatorial" world of simplicial sets.

8 Another example: algebraic theories

In this last section, we describe other examples of model structures. The objects we are interested in are a priori only some product-preserving functors, but they can be used to encode the axioms defining certain algebraic structures (an algebraic structure being defined as a set A with a collection of operations (maps from $A^n \to A$ for some $n \in \mathbb{N}$) and axioms, that is, identities these operations must satisfy), such as groups for example. From the free group on one generator and its successive coproducts with itself (namely, the free groups on n generators for $n \in \mathbb{N}$), we can recover a categorical definition of the notion of group. In this section, we follow the article "Algebraic theories in homotopy theory" by Badzioch (2002). Accordingly to the correspondence between topological spaces and simplicial sets seen in section (7), the two categories are identified in this article: simplicial sets are called "spaces" and the category $\mathbf{sSet}$ is denoted by **Spaces**. We will however keep the notation $\mathbf{sSet}$ for the category of simplicial sets, and the name "spaces" for *topological* spaces. This section is more descriptive and barely contains any proofs. All the results cited below without further reference are mentioned or proved in the article by Badzioch.

8.1 Algebraic theories and algebras

Let us give right away the main definitions:

Definition 8.1 (Algebraic theory)**.** An *algebraic theory* T is a small category with objects $\{T_n \mid n \in \mathbb{N}\}$, where T_0 is both initial and terminal, such that for all $n \in \mathbb{N}^*$ and $k \in \{1, \ldots, n\}$, there are maps $\pi_n^k : T_n \to T_1$ expressing T_n as the product $(T_1)^n$ of n copies of T_1 (but the category T may also contain other maps).

Definition 8.2 (Strict and homotopy algebras)**.** Let T be an algebraic theory.

- A *strict T-algebra* is a product-preserving functor $X : T \to \mathbf{sSet}$.
- A *homotopy T-algebra* is a functor $X : T \to \mathbf{sSet}$ preserving products up to weak equivalence: $X(T_0)$ is weakly equivalent to the point Δ^0 and for all $n \in \mathbb{N}^*$ the map

  $$(X(\pi_n^1), \ldots, X(\pi_n^n)) : X(T_n) \to X(T_1)^n$$

  is a weak equivalence (of simplicial sets, for the Quillen model structure).

Depending on the context, the category $\mathbf{sSet}$ in the definition above can be replaced by the category **Set** (for example, this is the definition given in the book "Algebraic theories" by Adámek *et al.* (2011)).

As previously advertised, algebraic theories and algebras allow us to encode axioms for some algebraic structures. Let us study the case of groups. Let **Gr** be the category of groups. Consider the full subcategory T^{op} of **Gr** with objects the free groups $F(n)$ on n generators for all $n \in \mathbb{N}$ (so the maps in this subcategory are all group homomorphisms between these objects). The group $F(0)$ is trivial, the group $F(1)$ is isomorphic to $\mathbb{Z}$ and $F(n) \cong \coprod_{i=1}^n F(1)$ (the coproduct is taken in the category of groups, it is the free product $*_{i=1}^n F(1)$). The elements of $F(n)$ can be described as the words on n letters, with powers in $\mathbb{Z}$. Let T be the opposite category to T^{op}. Since coproducts become products in the opposite category, T is an algebraic theory: $F(n) \cong \prod_{k=1}^n F(1)$ (product in T).

By definition, a group consists in a set G, together with maps $e : * \to G$ (identity element), $\mu : G \times G \to G$ (the group law) and $\bullet^{-1} : G \to G$ (inverses) such that, if $f : G \to *$ is the unique map from G to the terminal object in **Set**:

$$\begin{aligned} &\mu \circ (e \circ f, \mathrm{id}_G) = \mu \circ (\mathrm{id}_G, e \circ f) = \mathrm{id}_G && (e \text{ acts as the identity}) \\ &\mu \circ (\mathrm{id}_G \times \mu) = \mu \circ (\mu \times \mathrm{id}_G) && (\text{associativity}) \\ &\mu \circ (\mathrm{id}_G, \bullet^{-1}) = \mu \circ (\bullet^{-1}, \mathrm{id}_G) = e \circ f && (\text{inverses}) \end{aligned}$$

Consider the following maps in T^{op}: $e^{\mathrm{op}} : F(1) \to F(0)$ and $f^{\mathrm{op}} : F(0) \to F(1)$ the trivial homomorphisms, $\mu^{\mathrm{op}} : F(1) \to F(2)$, $a \mapsto ab$ and $(\bullet^{-1})^{\mathrm{op}} : F(1) \to F(1)$, $a \mapsto a^{-1}$ where a denotes a generator of $F(1)$, and a, b generators of $F(2)$. The opposite maps e, f, μ, and $\bullet^{-1}$ in T satisfy the identities above (for instance, $(\mathrm{id}_G^{\mathrm{op}} + (\bullet^{-1})^{\mathrm{op}}) \circ \mu^{\mathrm{op}} : F(1) \to F(1)$ maps the generator a to $f^{\mathrm{op}} \circ e^{\mathrm{op}}(a)$

(the identity element in $F(1)$) because $\mu^{\mathrm{op}}(a) = ab$ and $(\mathrm{id}_G^{\mathrm{op}} + (\bullet^{-1})^{\mathrm{op}})(ab) = aa^{-1} = f^{\mathrm{op}} \circ e^{\mathrm{op}}(a)$, and the other identities can be verified in a similar way). Therefore, if $X : T \to {}_{\mathbf{S}}\mathbf{Set}$ is a strict T-algebra, letting $G := X(F(1))$, the maps $X(e)$, $X(f)$, $X(\bullet^{-1})$, and $X(\mu) : X(F(2)) \cong G \times G \to G$ satisfy the identities above (by functoriality of X and the fact that it preserves products), so each set of simplices of $X(F(1))$ is endowed with a group structure (products in ${}_{\mathbf{S}}\mathbf{Set}$ are computed pointwise). Better than that, the degeneracies and faces become group homomorphisms. Indeed, the fact that $X(\mu)$ is a simplicial map means that it is a natural transformation between the functors $X(F(2))$ and $X(F(1))$. For example, for degeneracies we obtain a commutative diagram:

$$\begin{tikzcd}
X(F(1))^2_n \cong X(F(2))_n \arrow[r, "\mu_n"] \arrow[d, dashed, "X(F(2))(d_k)"] & X(F(1))_n \arrow[d, dashed, "X(F(1))(d_k)"] \\
X(F(1))^2_{n+1} \cong X(F(2))_{n+1} \arrow[r, "\mu_{n+1}"] & X(F(1))_{n+1}
\end{tikzcd}$$

for all $n \in N$ and $k \leq n+1$. Since μ_n and μ_{n+1} are the group laws on $X(F(1))_n$ and $X(F(1))_{n+1}$ respectively, this is exactly saying that the degeneracies in $X(F(1))$ are group homomorphisms.

Therefore, $X(F(1))$ can even be seen as a functor $\Delta^{\mathrm{op}} \to \mathbf{Gr}$ (what is called a *simplicial group*). In the same way, natural transformations between T-algebras become group homomorphisms. Replacing the category ${}_{\mathbf{S}}\mathbf{Set}$ with $\mathbf{Top}$, a product preserving functor $X : T \to \mathbf{Top}$ gives a *topological group* structure to $X(F(1))$, with continuous group law $X(\mu)$.

Conversely, if $(G, \mu, \bullet^{-1}, e)$ is a group, the functor $X : T \to {}_{\mathbf{S}}\mathbf{Set}$ mapping $F(n)$ to $\mathbf{Gr}(F(n), G)$ for all $n \in \mathbb{N}$ is a strict T-algebra: it preserves products because by the universal property of coproducts:

$$X(F(n)) = \mathbf{Gr}(F(n), G) \cong \mathbf{Gr}\left(\coprod_{i=1}^{n} F(1), G\right) \cong \prod_{i=1}^{n} \mathbf{Gr}(F(1), G) = (\mathbf{Gr}(F(1), G))^n = X(F(1))^n.$$

The group G is represented by $X(F(1))$ (any element of G determines a homomorphism $F(1) \to G$ and vice-versa). The set $\mathbf{Gr}(F(n), G)$ can be viewed as a simplical set by considering $\coprod_{\mathbf{Gr}(F(n),G)} \Delta^0$ (a "discrete" simplicial set with non-degenerate simplices only at order 0).

The same kind of argument can be performed for other algebraic structures, such as abelian groups, modules over a fixed ring, Lie algebras and so on.

For T any algebraic theory, let Alg_T be the full subcategory of $\mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set})$ with objects the product preserving functors (and morphisms the natural transformations between them). This is a category for strict T-algebras. In the next subsection, we will study model structures on categories of functors ("diagrams") and a category in which we can represent homotopy T-algebras.

8.2 Model structures on categories of functors

For the rest of this subsection, let T be a fixed algebraic theory.

8.2.1 The category of strict algebras

By the definition of Alg_T we gave above, it is a subcategory of $\mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set})$, so there is an inclusion functor $J_T : \mathrm{Alg}_T \to \mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set})$. The *adjoint functor theorem* provides sufficient conditions for a functor to have an adjoint. As Badzioch shows in his article, J_T satisfies the hypotheses of the theorem (the *solution set condition*, which basically asks for the existence of a *set* of maps $\{j_i : Y_i \to J_T(X_i)\}_{i \in I}$ in $\mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set})$ such that any map from an object of $\mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set})$ to an object in the image of J_T can be written as the composition of one of these maps and a map in the image of J_T: the maps from any object in $\mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set})$ to the image of J_T are well approximated by morphisms in the image of J_T and a "small" collection of other maps). So there is an adjunction:

$$\begin{tikzcd}
K_T : \mathrm{Fun}(T, {}_{\mathbf{S}}\mathbf{Set}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_T : J_T \arrow[l, shift left]
\end{tikzcd}$$

In his article, Badzioch uses a "lifting" lemma to transfer a model structure from one side of the adjunction to the other: the model category $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$ (defined in subsection (8.2.2) below) (more specifically, the right adjoint J_T) induces a model structure on Alg_T, where fibrations and weak equivalences are defined objectwise, and cofibrations are the maps having the LLP with respect to maps that are both fibrations and weak equivalences.

8.2.2 A category for homotopy algebras

In his article, Badzioch uses three different model structures on $\mathrm{Fun}(T, \mathbf{sSet})$:

- The first model structure seems pretty simple and natural: weak equivalences and fibrations are chosen to be objectwise weak equivalences and fibrations (if $F, F' : T \to \mathbf{sSet}$ and τ is a natural transformation between F and F', then τ is a weak equivalence, respectively a fibration, if and only if the simplicial map τ_{T_n} is a weak equivalence, respectively a fibration, for all $n \in \mathbb{N}$). Accordingly to proposition (2.9), cofibrations are defined as the maps having the LLP with respect to maps that are both fibrations and weak equivalences. The category $\mathrm{Fun}(T, \mathbf{sSet})$, when endowed with this model structure, will be denoted by $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$.

- The definition above is asymmetrical in terms of fibrations and cofibrations. Exchanging their roles, we can define a model structure on $\mathrm{Fun}(T, \mathbf{sSet})$ where weak equivalences and *cofibrations* are objectwise weak equivalences, respectively cofibrations, and fibrations are defined by the suitable lifting property. This time the model category we obtain is denoted by $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{cofib}}$.

- A third model structure, denoted by $\mathrm{LFun}(T, \mathbf{sSet})$, will allow us to represent homotopy algebras. It is defined as follows: weak equivalences are the *S-local equivalences* (see subsection (8.4.2) for the definition), cofibrations are the same as in $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$ (hence maps with the LLP with respect to objectwise weak equivalences and fibrations), and fibrations are defined as having the RLP with respect to maps that are both cofibrations and S-local equivalences.

The definition of $\mathrm{LFun}(T, \mathbf{sSet})$ above uses only the model category $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$. The category $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{cofib}}$ comes into play in the following proposition:

Proposition 8.3 (Characterization of S-local equivalences)**.** *A map $F : X \to X'$ in $\mathrm{LFun}(T, \mathbf{sSet})$ is an S-local equivalence if and only if for any homotopy T-algebra Z which is also fibrant in $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{cofib}}$, the simplicial map $f^* : \mathrm{Map}(X', Z) \to \mathrm{Map}(X, Z)$ induced by f is a weak equivalence (see subsection (8.4.2) for the definition of $\mathrm{Map}(-, -)$).*

The model structure $\mathrm{LFun}(T, \mathbf{sSet})$ is a particular case of a *(left) Bousfield localization*: under certain assumptions, a given model structure can be "enlarged" into a new one, with the same cofibrations but more weak equivalences, namely C-local equivalences, defined in a similar way as S-local equivalences, for another class of maps C. Fibrations are then defined, as usual, by the suitable lifting property.

Fibrant objects of $\mathrm{LFun}(T, \mathbf{sSet})$, represent homotopy T-algebras:

Proposition 8.4 (Characterization of homotopy T-algebras)**.** *A functor $X : T \to \mathbf{sSet}$ is fibrant as an object of $\mathrm{LFun}(T, \mathbf{sSet})$ if and only if it is both a homotopy algebra and a fibrant object of $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$.*

8.2.3 The Reedy model structure

In the proof of some results in his article, Badzioch uses the model category $\mathrm{Fun}(\Delta^{\mathrm{op}}, M)$ of simplicial objects in M, for several different categories M. The model structure he chooses on these categories is the *Reedy model structure*. Although this structure doesn't show up explicitly in this project, let us quickly discuss this additional example. We will compare it to the Quillen model structure on simplicial sets, and do another small example, which turns out to be useful to build *homotopy pushouts* (see Section 10 in the article by Dwyer and Spaliński (1995)). In this subsection, we follow the book by Hirschhorn (2003).

Definition 8.5 (Reedy category)**.** A *Reedy category* C is a small category, with a *degree* function $\deg : \mathrm{Ob}(C) \to \mathbb{N}$ assigning to each object a non-negative integer (in a more general version of the definition, degrees can be other ordinals), and two subcategories $\overrightarrow{C}$ and $\overleftarrow{C}$, containing only morphisms strictly raising the degree, respectively strictly lowering it (except for the identities). Moreover, every map f in C must have a unique factorization $f = \overrightarrow{f}\overleftarrow{f}$ with $\overrightarrow{f}$ in $\overrightarrow{C}$ and $\overleftarrow{f}$ in $\overleftarrow{C}$.

The degree orders the objects in the category and allows us to argue by induction on the degree when needed. The two examples we will be interested in are the category Δ^{op} and the category D with three objects a, b, c and two non-identity maps $a \xleftarrow{f} b \xrightarrow{g} c$, used in the construction of pushouts. The category D is a very simple Reedy category, we assign degree 0 to b and degree 1 to a and c. Then $\overrightarrow{D}$ is D itself, and $\overleftarrow{D}$ is the discrete category with objects $\{a, b, c\}$. Any given map factors as one identity map and the map itself. The category Δ^{op} is a Reedy category when we choose $\deg([n]) = n$ for all $n \in \mathbb{N}$, with $\overrightarrow{\Delta^{\mathrm{op}}}$ and $\overleftarrow{\Delta^{\mathrm{op}}}$ generated by degeneracies and faces respectively. The factorizations in the definition are given by the decomposition below remark (6.4) (the decomposition is made in Δ, so when passing to the opposite category, the codegeneracies written on the right hand side become degeneracies written on the left-hand side, similarly for (co)faces).

Let R be a Reedy category and C a model category. We would like to define a model structure on $\mathrm{Fun}(R, C)$. For this we need the following definitions:

Definition 8.6 (Latching category, object and map)**.** If b is an object in R, the *latching category* $\partial(\overrightarrow{R}{\downarrow}b)$ is the full subcategory of $\overrightarrow{R}{\downarrow}b$ (which is defined like $\Delta{\downarrow}X$ in definition (6.9)) containing all objects except the identity id_b. For X an object in $\mathrm{Fun}(R, C)$, there is a functor $X' : \partial(\overrightarrow{R}{\downarrow}b) \to C$ obtained by applying X to the source of the maps constituting the objects in $\partial(\overrightarrow{R}{\downarrow}b)$. Its colimit $L_bX := \mathrm{colim}_{\partial(\overrightarrow{R}\downarrow b)} X'$ is called the *latching object for* X and the map $\ell_{X,b} : L_bX \to X(b)$ induced by the images by X of all the maps constituting the objects in $\partial(\overrightarrow{R}{\downarrow}b)$ is called the *latching map*.

Definition 8.7 (Matching category, object and map)**.** If a is an object in R, the *matching category* $\partial(a{\downarrow}\overleftarrow{R})$ is the full subcategory of $a{\downarrow}\overleftarrow{R}$ (defined similarly to $\overleftarrow{R}{\downarrow}a$, except that we consider maps with source a instead of maps with target a) containing all objects except the identity id_a. Similarly there is a functor $X'' : \partial(a{\downarrow}\overleftarrow{R}) \to C$. Its limit $M_aX := \lim_{\partial(a\downarrow\overleftarrow{R})} X''$ is called the *matching object for* X and the map $m_{X,a} : X(a) \to M_aX$ induced by the images by X of all the maps constituting the objects in $\partial(a{\downarrow}\overleftarrow{R})$ is called the *matching map*.

Now we are ready to define the Reedy model structure:

Theorem 8.8 (The Reedy model structure)**.** *If R is a Reedy category and C a model category, the category* $\mathrm{Fun}(R, C)$ *is a model category with the definitions:*

- *A map $f : X \to Y$ is a weak equivalence if and only if $f_a : X(a) \to Y(a)$ is a weak equivalence in C for all objects a in R.*
- *A map $f : X \to Y$ is a cofibration if and only if for every object b in R the* relative latching map $r\ell_{f,b}$ *obtained by the universal property of pushouts:*

$$\begin{tikzcd}
L_bX \arrow[r, "\ell_{X,b}"] \arrow[d] & X(b) \arrow[d] \arrow[ddr, bend left, "f_b"] & \\
L_bY \arrow[r] \arrow[drr, bend right, "\ell_{Y,b}"'] & X(b) \amalg_{L_bX} L_bY \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "r\ell_{f,b}"] & \\
 & & Y(b)
\end{tikzcd}$$

is a cofibration in C (the map on the left hand side of the square is induced by f in the universal property of colimits for L_bX).

- *A map $f : X \to Y$ is a fibration if and only if for every object a in R the* relative matching map $rm_{f,a}$ *obtained by the universal property of pullbacks:*

$$\begin{tikzcd}
X(a) \arrow[dr, dashed, "rm_{f,a}"] \arrow[drr, bend left, "m_{X,a}"] \arrow[ddr, bend right, "f_a"'] & & \\
& Y(a) \times_{M_a Y} M_a X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & M_a X \arrow[d] \\
& Y(a) \arrow[r, "m_{Y,a}"'] & M_a Y
\end{tikzcd}$$

is a fibration in C (the map on the right hand side of the square is induced by f in the universal property of limits for $M_a Y$).

Since we defined the category of simplicial sets as $\mathbf{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ and Δ^{op} is a Reedy category, one could ask whether the Quillen model structure can be obtained as a Reedy model structure. The answer is no. As it turns out, there are exactly nine different model structures on the category **Set** (see for example the website of Antolín-Camarena (2010)), and weak equivalences for these structures do not suffice to ensure that objectwise weak equivalences in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ are exactly the simplicial maps whose geometric realization is a weak homotopy equivalence. In these structures, weak equivalences are either all maps, or the maps with non-empty domain and the identity of the empty set, or the bijections.

In the first and the second case, consider the simplicial map $f : S^1 \to *$, where S^1 is the "simplicial 1-sphere" introduced in subsection 7.1. The geometric realization of S^1 is homeomorphic to $\mathbb{S}^1$, hence it is not weakly equivalent to the point, even though objectwise the domain of f is not empty (there are at least degenerate simplices at all orders). In the situation where the weak equivalences in **Set** are the bijections, there is a counter-example in the other direction: consider the unique simplicial map $g : \Delta^1 \to *$. Since the geometric realizations of Δ^1 and $*$ are respectively the interval and the point, all their homotopy groups are trivial, so the geometric realization of g is necessarily a weak equivalence. However, g is not objectwise bijective: $* = \Delta^0$ has $|\Delta([1], [0])| = 1$ simplex of order 1, whereas Δ^1 has $|\Delta([1], [1])| = 3$ simplices of order 1. Hence the Quillen model structure on $\mathbf{sSet}$ is not a Reedy model structure.

Now we describe the Reedy model structure on $\mathrm{Fun}(D, C)$. An object X in this category is a diagram of the form $X(a) \xleftarrow{X(f)} X(b) \xrightarrow{X(g)} X(c)$ in C, and the maps between two objects X and X' correspond to triples of maps $h = (h_a, h_b, h_c)$ in C with a commutative diagram:

$$\begin{tikzcd}
X(a) \arrow[d, "h_a"'] & X(b) \arrow[l, "X(f)"'] \arrow[r, "X(g)"] \arrow[d, dashed, "h_b"] & X(c) \arrow[d, "h_c"] \\
X'(a) & X'(b) \arrow[l, "X'(f)"] \arrow[r, "X'(g)"'] & X'(c)
\end{tikzcd}$$

The theorem above tells us that h is a weak equivalence if and only if h_a, h_b, and h_c are weak equivalences in C. The latching categories for a, b, and c are given by $\{b \to a\}$, $\emptyset$ and $\{b \to c\}$ respectively. Then $L_a X = X(b)$, $L_b X = \emptyset$, and $L_c X = X(b)$. Hence h is a cofibration if and only if $r\ell_{h,b} = h_b : X(b) \amalg_\emptyset \emptyset = X(b) \to X'(b)$ is a cofibration and the two dotted arrows in the following diagram are cofibrations:

$$\begin{tikzcd}
& X(a) \arrow[d] \arrow[dl, bend right, "h_a"'] & X(b) \arrow[l, "X(f)"'] \arrow[r, "X(g)"] \arrow[d, dashed, "h_b"] & X(c) \arrow[d] \arrow[dr, bend left, "h_c"] & \\
& X(a) \amalg_{X(b)} X'(b) \arrow[dl, dashed] & X'(b) \arrow[l] \arrow[r] \arrow[dll, bend left, "X'(f)"] \arrow[drr, bend right, "X'(g)"'] & X(c) \amalg_{X(b)} X'(b) \arrow[dr, dashed] & \\
X'(a) & & & & X'(c)
\end{tikzcd}$$

For fibrations, the matching categories of a, b, and c are all empty since there are no non-identity maps lowering degree in D. Their matching object is the terminal object. The relative matching maps are h_a, h_b, and h_c respectively: fibrations are pointwise fibrations.

8.3 A Quillen equivalence

Let T be an algebraic theory. The main results presented in the article by Badzioch (2002) are:

Theorem 8.9 (Comparing homotopy and strict algebras)**.** *There is a Quillen adjunction and equivalence:*

$$K_T : \mathrm{LFun}(T, \mathbf{sSet}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Alg}_T : J_T$$

Theorem 8.10 ("Rigidifying" result)**.** *Every homotopy T-algebra is weakly equivalent (by an objectwise weak equivalence) to a strict T-algebra.*

If we admit that the pair of functors in theorem (8.9) form a Quillen adjunction, the rest of the two statements above follows from the following propositions, which we again admit without proof:

Proposition 8.11 (η_X is a weak equivalence)**.** *Let $\eta : \mathrm{id}_{\mathrm{LFun}(T,\mathbf{sSet})} \to J_T \circ K_T$ be the unit of the Quillen adjunction in theorem (8.9). Then, for any cofibrant object X in $\mathrm{LFun}(T, \mathbf{sSet})$, the map $\eta_X : X \to J_T K_T(X)$ is an S-local equivalence (i.e. a weak equivalence in $\mathrm{LFun}(T, \mathbf{sSet})$).*

Proposition 8.12 (S-local and objectwise weak equivalences)**.** *If both X and X' are homotopy T-algebras and $f : X \to X'$ is an S-local equivalence between them (in $\mathrm{LFun}(T, \mathbf{sSet})$), then f is an objectwise weak equivalence (in $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$).*

Using these propositions, we can show that the additional condition in theorem (4.4) holds, turning the Quillen adjunction in theorem (8.9) into a Quillen equivalence. We have to show that the bijection $\mathrm{Alg}_T(K_T(X), Y) \to \mathrm{LFun}(T, \mathbf{sSet})(X, J_T(Y))$ induced by the adjunction preserves and creates weak equivalences, for any cofibrant object X in $\mathrm{LFun}(T, \mathbf{sSet})$ and fibrant object Y in Alg_T.

Let $f : K_T(X) \to Y$ be a map in Alg_T. Then, we have the identity $f^\sharp = f \circ \eta_X$ (recall that J_T is just an inclusion). Since X is cofibrant, η_X is an S-local equivalence by proposition (8.11).

Assume first that f is an objectwise weak equivalence. Then it is also an S-local equivalence when seen as a map in $\mathrm{LFun}(T, \mathbf{sSet})$: indeed, the homotopy function complex introduced in subsection (8.4.2) preserves weak equivalences. Hence, by composition, $f^\sharp$ is also an S-local equivalence. Conversely, if we now assume that $f^\sharp$ is an S-local equivalence, by the 2 of 3 rule in the model category $\mathrm{LFun}(T, \mathbf{sSet})$, f is an S-local equivalence. Since $K_T(X)$ and Y are both strict T-algebras, they are in particular homotopy T-algebras, so by proposition (8.12), f is an objectwise weak equivalence, and we are done.

We can also prove theorem (8.10): let X be a homotopy T-algebra. Consider it as an object in $\mathrm{LFun}(T, \mathbf{sSet})$. Let Y be a bifibrant replacement of X in $\mathrm{LFun}(T, \mathbf{sSet})$, weakly equivalent to X by an S-local equivalence τ (it is always possible to find such a replacement in a model category). By proposition (8.11), Y is weakly equivalent in $\mathrm{LFun}(T, \mathbf{sSet})$ to the strict T-algebra $J_T K_T(Y)$. By composition, we have an S-local equivalence between X and the strict T-algebra $J_T K_T(Y)$. Both X and $J_T K_T(Y)$ are homotopy T-algebras. Therefore, by proposition (8.12), they are actually (objectwise) weakly equivalent in $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$.

8.4 Appendix: some more definitions

8.4.1 Simplicial model categories

Following Hirschhorn (2003), we have the definition:

Definition 8.13 (Simplicial model category)**.** A *simplicial model category* C is a locally small model category with, for every objects W, X, Y, and Z in C:

(i) a simplicial set $\mathrm{Hom}(X, Y)$.

(ii) a simplicial map (by analogy to the composition)

$$c_{X,Y,Z} : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z).$$

(iii) a simplical map $i_X : * \to \mathrm{Hom}(X, X)$.

(iv) a bijection $\mathrm{Hom}(X, Y)_0 \cong C(X, Y)$ that commutes with the composition.

(v) associativity of the composition, and the maps i_X act as identities for composition, namely:

$$\begin{aligned}
&c_{W,X,Z} \circ (c_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}(W,X)}) = c_{W,Y,Z} \circ (\mathrm{id}_{\mathrm{Hom}(Y,Z)} \times c_{W,Y,Z}) \\
&c_{X,Y,Y} \circ (i_Y, \mathrm{id}_{\mathrm{Hom}(X,Y)}) = \mathrm{id}_{\mathrm{Hom}(X,Y)} \\
&c_{X,X,Y} \circ (\mathrm{id}_{\mathrm{Hom}(X,Y)}, i_X) = \mathrm{id}_{\mathrm{Hom}(X,Y)}
\end{aligned}$$

(vi) for every simplicial set K, objects $X \otimes K$ and Y^K in C such that there are isomorphisms of simplicial sets, natural in all variables:

$$\mathrm{Hom}(X \otimes K, Y) \cong \mathrm{Hom}(X, Y^K) \cong \mathrm{Map}(K, \mathrm{Hom}(X,Y))$$

where $\mathrm{Map}(A,B)$ for two simplicial sets A and B denotes the simplicial set with k-simplices $\mathrm{Map}(A,B)_k = \mathbf{sSet}(A \times \Delta^k, B)$ with faces and degeneracies acting exclusively on Δ^k, like the standard faces and degeneracies in Δ^{op}, for all $k \in \mathbb{N}$.

(vii) If $\iota : W \hookrightarrow Z$ is a cofibration in C and $p : X \twoheadrightarrow Y$ is a fibration in C, the simplicial map:

$$\mathrm{Hom}(Z,X) \xrightarrow{(\iota^*, p_*)} \mathrm{Hom}(W,X) \times_{\mathrm{Hom}(W,Y)} \mathrm{Hom}(Z,Y),$$

induced by the universal property of the pullback, is a fibration (with p_* and ι^* the maps induced by p and ι respectively by "composition", as we would proceed with usual Hom functors). If additionally either ι or p is a weak equivalence then (ι^*, p_*) must be a weak equivalence too.

We ask for the category to be locally small because in point (iv), $\mathrm{Hom}(X,Y)_0$ is a *set*. The point (iv) tells us that we can think about $\mathrm{Hom}(X,Y)$ as "enriching" $C(X,Y)$ with a simplicial set structure. In this situation, the maps in point (ii) represent the composition and the maps in point (iii) the identities. The idea is to generalize the operations we can do with simplicial sets to other model categories. The object $X \otimes K$ generalizes the product of simplicial sets we could build if X was a simplicial set itself, and the object Y^K generalizes the simplicial set $\mathrm{Map}(K,Y)$, defined at point (vi) if Y is a simplicial set. Actually, these definitions, together with $\mathrm{Hom}(A,B) = \mathrm{Map}(A,B)$ for all simplicial sets A and B, turn $\mathbf{sSet}$ (with the Quillen model structure) into a simplicial model category. For more details, see Section 9.1 in the book by Hirschhorn (2003) or Sections II.1-II.3 in the book by Quillen (1967).

All three model categories listed in subsection (8.2.2) are simplicial model categories. The simplicial sets $\mathrm{Hom}(-,-)$ are defined as the function complexes of subsection (8.4.2). If $F : T \to \mathbf{sSet}$ is a functor and X is a simplicial set, we have $(F \otimes X)(T_n) = F(T_n) \times X$ for all $n \in \mathbb{N}$ (the symbol $\otimes$ on the left hand side is the one in the definition of a simplicial model category, whereas the product symbol on the right hand side denotes the usual categorical product). The object X^K is defined as the functor $\mathrm{Map}(K,-) \circ X : T \to \mathbf{sSet}$.

8.4.2 Function complexes, homotopy function complexes and S-local equivalences

Badzioch uses in his article the following constructions:

If C is a simplicial model category, for any two objects X and Y in C, there is a simplicial set $\mathrm{Map}(X,Y)$ with k-simplices $\mathrm{Map}(X,Y)_k = C(X \otimes \Delta^k, Y)$ for all $k \in \mathbb{N}$, generalizing the same construction we saw in $\mathbf{sSet}$. This is called the *simplicial function complex for X and Y*. There also exist a *homotopy function complex*: a simplicial set $\mathrm{RMap}(X,Y)$ whose path components (see definition (7.2)) are in bijection with the maps between X and Y in the homotopy category of C, in other words, $\pi_0\mathrm{RMap}(X,Y) \cong \mathrm{Ho}(C)(X,Y)$. It turns out that, in the situation where X is cofibrant and Y fibrant, the two complexes are weakly equivalent (as simplicial sets). Another useful property is that the homotopy function complex preserves weak equivalences: if X and Y are weakly equivalent in C to X' and Y' respectively, then $\mathrm{RMap}(X,Y)$ is weakly equivalent as a simplicial set to $\mathrm{RMap}(X',Y')$. Also, any $f : X \to X'$ induces a simplicial map $f^* : \mathrm{RMap}(X',Y) \to \mathrm{RMap}(X,Y)$ (and similarly on the second coordinate).

We can now define S-local equivalences in the category $\mathrm{LFun}(T, \mathbf{sSet})$.

Definition 8.14 (The set S)**.** For each $n \in \mathbb{N}$, there is a functor F_n in Alg_T with $F_n(T_m) = T(T_n, T_m)$ (actually, $T(T_n, T_m)$ is the set of 0-simplices, and all other simplices are their iterated degeneracies) for all $m \in \mathbb{N}$. When $n \neq 0$, the projections π_n^k in T with $1 \leq k \leq n$ induce maps

$$p_n : \coprod_{i=1}^{n} F_1 = T((T^1)^n, -) \longrightarrow F_n = T(T_n, -).$$

Let $p_0 : \emptyset \to F_0$ ($\emptyset$ also denotes the empty T-algebra). Then, we define $S := \{p_n \mid n \in \mathbb{N}\}$.

Definition 8.15 (S-local objects)**.** An *S-local object* is a fibrant object Z in $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$ such that the induced simplicial map of homotopy function complexes $p_n^* : \mathrm{RMap}(F_n, Z) \to \mathrm{RMap}(\coprod_{i=1}^n F_1, Z)$ is a weak equivalence for all $n \in \mathbb{N}$.

Definition 8.16 (S-local equivalences)**.** A map $f : X \to Y$ in $\mathrm{Fun}(T, \mathbf{sSet})_{\mathrm{fib}}$ is an *S-local equivalence* if the induced simplicial map $f^* : \mathrm{RMap}(Y, Z) \to \mathrm{RMap}(X, Z)$ is a weak equivalence for any S-local object Z.

In particular, the maps in S are S-local equivalences.

9 Conclusion

The main goal of this semester project was to get to know the concept of model categories. We studied the axioms of model structures, and some of their consequences, and built the homotopy category in two different ways. Finally, we showed how one can compare the homotopy categories of two different model categories, and applied this to the example of the Quillen model structure on topological spaces and simplicial sets. On our journey, we met interesting constructions and arguments, from simple but clever use of the axioms in lemma (4.5) to arguments inspired by topological constructions in proposition 3.7, as well as the (transfinite) small object argument, among others.

Some authors give a slightly different definition of a model category, for example, Hovey (1999) asks in his book for the *functoriality* of the factorizations in axiom MC5. One of the advantages of the small object argument is that it actually produces functorial factorizations, so the cases of **Top** and $_{\mathbf{S}}\mathbf{Set}$ satisfy this stronger definition too.

We have also seen that the axioms defining some algebraic structures can be encoded in algebraic theories. "Realizations" of these theories, namely strict algebras, take the form of product-preserving functors. Playing on different model structures on categories of diagrams, one can obtain a "rigidifying result" (as Badzioch calls it): homotopy algebras can be turned into strict algebras, up to (objectwise) weak equivalence.

This was only a brief look at the world of model categories. With more time, one could also study a model structure on cochain complexes of modules over a ring, or the general notion of simplicial objects: we defined the category of simplicial sets as $_{\mathbf{S}}\mathbf{Set} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, and in a similar way, for a category C, there is a category of *simplicial objects in* C, namely $\mathrm{Fun}(\Delta^{\mathrm{op}}, C)$. For example, if $C = \mathbf{Gr}$ is the category of groups, we talk about *simplicial groups*. The constructions behind the proof of theorem (7.1) would also have been interesting extensions to this project: the proof uses the long exact homotopy sequences for both simplicial sets and topological spaces, together with the machinery of anodyne extensions (which correspond to acyclic cofibrations in $_{\mathbf{S}}\mathbf{Set}$ and are basically defined as the class of maps obtained by the operations of proposition (2.10) on the "horn-inclusions" of $\partial\Delta^n$ minus one face into Δ^n), and path and loop spaces.

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