The real realization of the motivic spectrum ko

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Introduction

A short dictionary giving a rough explanation of words in brown can be found below.

Motivic homotopy theory generalizes tools from topology and classical homotopy theory to algebraic geometry. Instead of studying topological spaces in a homotopy invariant way (the interval is equivalent to the point), it studies schemes with the affine line $\mathbb{A}^1$ made equivalent to the point. This yields the category of motivic spaces. As an analog to the category of topological spectra, one can build a category of motivic spectra. Working over $\mathbb{R}$, to any scheme one can associate the space of its real points. Similarly, to a motivic spectrum E is associated a topological spectrum $r_{\mathbb{R}}(E)$: its *real realization*, which can provide some information or intuition about E.

Goal: identify $r_{\mathbb{R}}(\mathsf{ko})$, where ko is a refinement (*the very effective cover*) of the motivic spectrum KO, which represents a cohomology theory for schemes involving vector bundles with a symmetric form. Let us explore in greater details what this means.

Motivic spaces

Let $\mathsf{Sm}_{\mathbb{R}}$ be the category of smooth schemes of finite type over $\mathbb{R}$ (think: very reasonable schemes).
Definition: A motivic space is a functor $\mathcal{F} : \mathsf{Sm}_{\mathbb{R}}^{\mathsf{op}} \to \mathsf{Spc}$ where Spc is the category of spaces (namely a presheaf of spaces on $\mathsf{Sm}_{\mathbb{R}}$) such that:

1. It is $\mathbb{A}^1$-invariant: for all $X \in \mathsf{Sm}_{\mathbb{R}}$, the map $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$ induced by the projection is an equivalence (i.e. $\mathcal{F}$ does not see the difference between $\mathbb{A}^1$ and the point).
2. It is a Nisnevich sheaf, i.e. it is determined by its local behaviour with respect to the Nisnevich topology (coarser than the étale topology but finer than the Zariski one).

Upshot: any smooth scheme or any space yields a motivic space, but there are many other ones. Any presheaf also yields one, because there is a localization functor $\mathsf{L}_{\mathsf{mot}}$ turning any presheaf into one satisfying 1. and 2..

Analogy: Replacing smooth schemes by smooth manifolds, $\mathbb{A}^1$ by the real line, and the Nisnevich topology by the Euclidean topology, this construction yields the usual homotopy category of spaces.

Motivic spectra and connectivity

The category of pointed motivic spaces $\mathsf{Spc}(\mathbb{R})_*$ has a commutative multiplicative structure (i.e. it is *symmetric monoidal*) given by the smash product $\wedge$. Here, $X \wedge Y$ is formed by collapsing the subspace $(X \times *) \cup (* \times Y)$ in $X \times Y$ to a point. Topological spectra arise from pointed spaces Spc_* by making the circle $\mathbb{S}^1$ invertible for the smash product. In motivic homotopy, there are two circles: the usual space $\mathcal{S}^1$ and the scheme $\mathbb{G}_m = \mathsf{Spec}(\mathbb{R}[t, t^{-1}])$. Motivic spectra are constructed by inverting both, or equivalently, inverting their smash product (equivalent in $\mathsf{Spc}(\mathbb{R})_*$ to the scheme $\mathbb{P}^1$):

Definition: The category of motivic ($\mathbb{P}^1$-)spectra is the universal symmetric monoidal category $\mathsf{SH}(\mathbb{R})$ with a symmetric monoidal functor $\Sigma^\infty_{\mathbb{P}^1} : \mathsf{Spc}(\mathbb{R})_* \to \mathsf{SH}(\mathbb{R})$ sending $\mathcal{S}^1 \wedge \mathbb{G}_m$ to an invertible object.

In analogy to topological spectra, a motivic spectrum can be viewed as a sequence $(X_i)_{i \geq 0} \subseteq \mathsf{Spc}(\mathbb{R})_*$ with chosen equivalences $X_i \simeq \Omega_{\mathbb{P}^1} X_{i+1}$ for all $i \geq 0$, where $\Omega_{\mathbb{P}^1}$ is right adjoint to $\Sigma_{\mathbb{P}^1} = \mathbb{P}^1 \wedge -$.

Spectra have stable homotopy groups indexed by $\mathbb{Z}$, and are called connective when these groups vanish in negative degrees. The corresponding notion for motivic spectra yields bigraded objects, because we have two circles. In this setting, we need a notion of connectivity with respect to both motivic circles: this is called very effectiveness.

From motivic to topological: real realization

Real realization extends the construction of taking the real points of a scheme (over $\mathbb{R}$). It is a functor $r_{\mathbb{R}} : \mathsf{SH}(\mathbb{R}) \to \mathsf{Sp}$ from motivic spectra to topological spectra, with the following properties:

- it preserves colimits and finite products;
- it preserves the multiplicative structures (it is symmetric monoidal);
- for a motivic spectrum that comes from a scheme (via the infinite suspension functor $\Sigma^\infty_{\mathbb{P}^1}$), it just takes the suspension spectrum of the space of its real points;
- it maps very effective motivic spectra to connective topological spectra.

First examples (omitting the infinite suspension functors on both sides):

$$r_{\mathbb{R}}(\mathbb{A}^1) \simeq \mathbb{R} \simeq *, \qquad r_{\mathbb{R}}(\mathcal{S}^1) \simeq \mathbb{S}^1, \qquad r_{\mathbb{R}}(\mathbb{G}_m) \simeq \mathbb{R}^\times \simeq \mathbb{S}^0, \qquad r_{\mathbb{R}}(\mathbb{P}^1) \simeq r_{\mathbb{R}}(\mathcal{S}^1) \wedge r_{\mathbb{R}}(\mathbb{G}_m) \simeq \mathbb{S}^1.$$

Hermitian K-theory

Hermitian K-theory is a cohomology theory for schemes: it associates to every (affine) scheme X a sequence of abelian groups $\{\mathsf{KO}_n(X)\}_{n \in \mathbb{Z}}$ constructed as follows. The collection of algebraic vector bundles over X (schemes with a map to X, which looks Zariski-locally like the projection $X \times \mathbb{A}^n \to X$) with a symmetric form (compatible ones on the $\mathbb{A}^n$'s) forms a monoid with respect to the direct sum. By adding formal inverses, we can make this into a group $\mathsf{KO}_0(X)$ (this is called group completion). One can reproduce this construction to obtain a topological monoid, and then group-complete it; the homotopy groups of the space this yields are the groups $\{\mathsf{KO}_n(X)\}_n$.

Hermitian K-theory is represented by a motivic spectrum KO: for $X \in \mathsf{Sm}_{\mathbb{R}}$ affine, the homotopy classes of maps $X_+ \wedge \mathbb{G}_m^{\wedge q} \wedge \mathcal{S}^{p-q} \to \mathsf{KO}$ in $\mathsf{SH}(\mathbb{R})$ recover $\mathsf{KO}_{p-2q}(X)$ for all $p, q \in \mathbb{Z}$. Then, the tensor product of vector bundles induces a multiplication $\mathsf{KO} \wedge \mathsf{KO} \to \mathsf{KO}$ and a unit $\mathcal{S}^0 \to \mathsf{KO}$ satisfying the axioms of a commutative monoid up to homotopy: it makes KO into a motivic $\mathcal{E}_\infty$-ring.

Our goal and some challenges of the proof

- Goal: identify $r_{\mathbb{R}}(\mathsf{ko})$, where ko is the universal very effective spectrum approximating KO (its very effective cover). This can mean several things.
- Indeed, $r_{\mathbb{R}}\mathsf{ko}$ is a spectrum, but it also has the structure of an $\mathcal{E}_\infty$-ring, or $\mathcal{E}_1$-ring (if we forget about all the data related to commutativity) inherited from KO. We can try to identify $r_{\mathbb{R}}\mathsf{ko}$ at these three different levels. The $\mathcal{E}_1$- and $\mathcal{E}_\infty$-structures are usually subtle to deal with, because they encapsulate a lot of data. For example, any two $\mathcal{E}_1$- or $\mathcal{E}_\infty$-rings with a map of spectra between them, inducing isomorphisms of graded rings in homotopy, are equivalent as spectra, but the $\mathcal{E}_1$- or $\mathcal{E}_\infty$-structures do not always agree.
- We work with $\mathcal{E}_1$-rings, because in this case we have a good description of some free objects (which basically means that it is easy to produce maps of $\mathcal{E}_1$-rings out of them).
- Another challenge is that the construction of the functor $r_{\mathbb{R}}$ is not really explicit; and the way it interacts with very effective covers is unclear. For example, $r_{\mathbb{R}}\mathsf{KO}$ has already been computed in [BH20], but not $r_{\mathbb{R}}\mathsf{ko}$. Moreover, although we know the pointed motivic spaces forming the sequence that defines KO as a motivic spectrum, we cannot tell what they become for ko.

Results

Proposition. *There is an isomorphism of graded rings $\pi_*(r_{\mathbb{R}}\mathsf{ko}) \cong \mathbb{Z}[x]$, where x has degree 4.*

Theorem A. *There is a pullback square of $\mathcal{E}_1$-rings (and thus also of spectra):*

$$\begin{array}{ccc} r_{\mathbb{R}}(\mathsf{ko}) & \xrightarrow{x \mapsto \beta_4/2} & \mathsf{KO}^{\mathsf{top}}[1/2]_{\geq 0} \\ \downarrow{\scriptstyle x \mapsto t^4} & \lrcorner & \downarrow{\scriptstyle \beta_4/2 \mapsto t^4} \\ \mathsf{H}\mathbb{Z}_{(2)}[t^4] & \xrightarrow[t^4 \mapsto t^4]{} & \mathsf{H}\mathbb{Q}[t^4] \end{array}$$

what happens to $r_{\mathbb{R}}(\mathsf{ko})$ if we make 2 invertible — $\mathsf{KO}^{\mathsf{top}}[1/2]_{\geq 0}$

what happens to $r_{\mathbb{R}}(\mathsf{ko})$ if we localize at (2) (make all odd primes invertible) — $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$

what happens to $r_{\mathbb{R}}(\mathsf{ko})$ if we take rationalization (make all primes invertible) — $\mathsf{H}\mathbb{Q}[t^4]$

and $r_{\mathbb{R}}(\mathsf{ko})$ is equivalent as an $\mathcal{E}_1$-ring to $\mathsf{L}(\mathbb{R})_{\geq 0}$, the connective cover of the L-theory spectrum of $\mathbb{R}$.

Here, for A a ring and HA its Eilenberg-MacLane spectrum, $\mathsf{HA}[t^4]$ is the free $\mathcal{E}_1$-HA-algebra on one generator in degree 4. It is an HA-module, with homotopy ring $\pi_*(\mathsf{HA}[t^4]) \cong \mathsf{A}[t^4]$. The arrows are labeled with the description of the maps they induce on π_4, with $\beta_4 \in \pi_4\mathsf{KO}^{\mathsf{top}} \cong \mathbb{Z}$ a generator.

Idea: The square in Theorem A gives a decomposition of $r_{\mathbb{R}}(\mathsf{ko})$ in terms of simpler pieces. What we had to do was to identify these pieces and the way they fit together more explicitly. We can then deduce that $r_{\mathbb{R}}(\mathsf{ko})$ is equivalent to $L(\mathbb{R})_{\geq 0}$, because the latter admits a similar square ([HLN21]): both are made of the same pieces, assembled in the same way.

To prove the previous theorem, we also had to show:

Theorem B. *The motivic and topological Thom spectrum functors correspond to each other, as symmetric monoidal functors, under real realization. In particular, there are equivalences of $\mathcal{E}_\infty$-rings:*

$$r_{\mathbb{R}}(\mathsf{MGL}) \simeq \mathsf{MO}, \qquad r_{\mathbb{R}}(\mathsf{MSL}) \simeq \mathsf{MSO}, \qquad r_{\mathbb{R}}(\mathsf{MSp}) \simeq \mathsf{MU}.$$

Concepts and ideas used in the proof

- Let us note that all the constructions described before are performed in an ∞-categorical setting.
- To compute the homotopy groups of $r_{\mathbb{R}}(\mathsf{ko})$, one uses the Wood cofiber sequence $\mathbb{G}_m \wedge \mathsf{ko} \to \mathsf{ko} \to \mathsf{kgl}$ ([ARØ20]) and the long exact sequence in homotopy it induces after real realization. To make some arguments easier, we cut ko into "slices" which we analyze individually. These slices have been computed in [Bac17]; they are the successive subquotients in some sort of Postnikov tower for ko, where the notion of very effectiveness plays the role of connectivity. When identifying their real realization we encounter several interesting concepts such as computations in the motivic Steenrod algebra and Milnor's conjecture (nowadays proved).
- In the identification of the top right corner of the square, i.e. when 2 is made invertible, is involved Brumfiel's theorem: the signature homomorphism from Balmer-Witt theory (related to ko) to real topological K-theory (represented by $\mathsf{KO}^{\mathsf{top}}$) is an equivalence after inverting 2.
- Seeing what happens when we invert odd primes is harder, and is the reason why we identify the $\mathcal{E}_1$-ring structure and not the $\mathcal{E}_\infty$ one. We make a map of $\mathcal{E}_1$-rings from $\mathsf{H}\mathbb{Z}_{(2)}[t^4]$ to the object we want to identify, and then show it is an equivalence. Since the source is a free object, we begin by finding a map of $\mathcal{E}_1$-rings with source $\mathsf{H}\mathbb{Z}_{(2)}$ instead. As an intermediate step, we first map to the real realization of a motivic Thom spectrum $r_{\mathbb{R}}\mathsf{MSL}_{(2)}$, which is a topological Thom spectrum $\mathsf{MSO}_{(2)}$ by Theorem B. Then a map of $\mathcal{E}_1$-rings $\mathsf{H}\mathbb{Z}_{(2)} \to \mathsf{MSO}_{(2)}$ is obtained using the topological Thom spectrum functor; since both sides can be written as Thom spectra.
- Studying the maps in the square requires some work. In particular, it involves computing the cardinalities of certain homotopy groups, using multiplicative spectral sequences, where the differentials satisfy the Leibniz rule. Their multiplicative structure is again induced by that of KO.

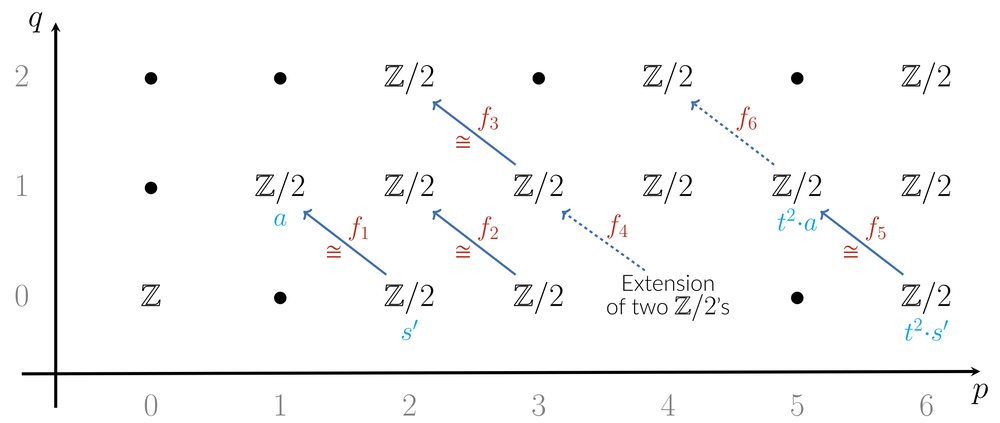

Figure 1. A spectral sequence we used

Dictionary

- Cohomology theory: associates to every space a sequence of abelian groups, in a way that is invariant under homotopy equivalences and has good exactness properties.
- Up to homotopy: the axioms required, written under the form of diagrams, commute not on the nose but up to a specified identification (i.e. homotopy) between the different composites.
- $\mathcal{E}_1$-algebra, resp. $\mathcal{E}_\infty$-algebra: object in a category with the structure of a monoid (resp. commutative monoid) "up to homotopy". $\mathcal{E}_1$-ring, resp. $\mathcal{E}_\infty$-ring: an $\mathcal{E}_1$-algebra, resp. $\mathcal{E}_\infty$-algebra in a category of spectra.
- Schemes: the basic objects of algebraic geometry, obtained by glueing together algebraic varieties defined as the vanishing locus of polynomial equations.
- Spectrum: sequence of spaces such that every term is the loop space of the following one. Spectra represent cohomology theories. The category of spectra enlarges that of spaces, with the defining property that the topological suspension functor becomes an equivalence from the category of spectra to itself.

References and contact

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Picture below: "Cosmic cliffs" in the Carina Nebula; James Webb telescope; NASA, ESA, CSA, STScI.

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